

Anisotropic cosmological solutions in massive vector theories

Lavinia Heisenberg¹, Ryotaro Kase², and Shinji Tsujikawa²

¹*Institute for Theoretical Studies, ETH Zurich, Clausiusstrasse 47, 8092 Zurich, Switzerland*

²*Department of Physics, Faculty of Science, Tokyo University of Science,
1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan*

(Dated: July 13, 2016)

In beyond-generalized Proca theories including the extension to theories higher than second order, we study the role of a spatial component v of a massive vector field on the anisotropic cosmological background. We show that, as in the case of the isotropic cosmological background, there is no additional ghostly degrees of freedom associated with the Ostrogradski instability. In second-order generalized Proca theories we find the existence of anisotropic solutions on which the ratio between the anisotropic expansion rate Σ and the isotropic expansion rate H remains nearly constant in the radiation-dominated epoch. In the regime where Σ/H is constant, the spatial vector component v works as a dark radiation with the equation of state close to $1/3$. During the matter era, the ratio Σ/H decreases with the decrease of v . As long as the conditions $|\Sigma| \ll H$ and $v^2 \ll \phi^2$ are satisfied around the onset of late-time cosmic acceleration, where ϕ is the temporal vector component, we find that the solutions approach the isotropic de Sitter fixed point ($\Sigma = 0 = v$) in accordance with the cosmic no-hair conjecture. In the presence of v and Σ the early evolution of the dark energy equation of state w_{DE} in the radiation era is different from that in the isotropic case, but the approach to the isotropic value $w_{\text{DE}}^{(\text{iso})}$ typically occurs at redshifts z much larger than 1. Thus, apart from the existence of dark radiation, the anisotropic cosmological dynamics at low redshifts is similar to that in isotropic generalized Proca theories. In beyond-generalized Proca theories the only consistent solution to avoid the divergence of a determinant of the dynamical system corresponds to $v = 0$, so Σ always decreases in time.

I. INTRODUCTION

Cosmology is facing a challenge of revealing the origin of unknown dark components dominating the present Universe. The standard cosmological model introduces a pure cosmological constant Λ to the field equations of General Relativity (GR), and additionally a non-baryonic dark matter component in the form of non-relativistic particles, known as cold dark matter. This rather simple model is overall consistent with the tests at cosmological scales by the Cosmic Microwave Background (CMB) anisotropies [1], by the observed matter distribution in large-scale structures [2], and by the type Ia supernovae [3].

In the prevailing view the cosmological constant should arise from the vacuum energy density, which can be estimated by using techniques of the standard quantum field theory. Lamentably, the theoretically estimated value of vacuum energy is vastly larger than the observed dark energy scale. This is known as the cosmological constant problem [4]. In this context, infrared modifications of gravity have been widely studied in the hope to screen the cosmological constant via a high-pass filter, naturally arising in higher dimensional set-ups, in massive gravity [5] or non-local theories [6].

On the same footing as tackling the cosmological constant problem, infrared modifications of gravity may yield accelerated expansion of the Universe due to the presence of new physical degrees of freedom, providing an alternative framework for dark energy [7]. The simplest realization is in form of an additional scalar degree of freedom, that couples minimally to gravity [8]. Richer phenomenology can be achieved by allowing for self-interactions of the scalar field or non-minimal couplings to gravity [9]. Extensively studied classes are the Galileon [10, 11] and Horndeski [12] theories, whose constructions rely on the condition of keeping the equations of motion up to second order [13]. Easing this restriction allows for interactions with higher-order equations of motion, but it is still possible to construct theories that propagate only one scalar degree of freedom (DOF)—Gleyzes-Langlois-Piazza-Vernizzi (GLPV) theories [14] (see also Refs. [15] for the discussion of the propagating DOF). These beyond-Horndeski interactions generalize the previous ones and offer richer phenomenology [16, 17].

Even if the extension in the form of an additional scalar field is the simplest and most explored one, the inclusion of additional vector fields has been taken into consideration as well [18–22]. The Maxwell field with the standard kinetic term is a gauge-invariant vector field with two propagating transverse modes. The attempt to construct derivative self-interactions for a massless, Lorentz-invariant vector field yielded a no-go result [23], but this can be overcome by breaking the gauge invariance. The simplest way of breaking gauge invariance is to include a mass term for the vector field, which is known as the Proca field. The inclusion of allowed derivative self-interactions and non-minimal couplings to gravity results in the generalized Proca theories with second-order equations of motion [24–27]. As in the original Proca theory, there are still the three physical DOFs, one of them being the longitudinal mode and the

other two corresponding to the standard transverse modes (besides two tensor polarizations). The phenomenology of (sub classes of) generalized Proca theories has been extensively studied in Refs. [28–31].

As in the GLPV extension of scalar Horndeski theories, relaxing the condition of second-order equations of motions in generalized Proca theories allows us to construct new higher-order vector-tensor interactions [32] without the Ostrogradski instability [33]. In Ref. [32] it was shown that, even in the presence of interactions beyond the domain of generalized Proca theories, there are no additional DOFs associated with the Ostrogradski ghost on the isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) background. In fact, the Hamiltonian \mathcal{H} is equivalent to 0 due to the existence of a constraint that removes the appearance of a ghostly DOF.

In Refs. [30, 31] the cosmology of generalized Proca theories was studied by introducing the temporal component $\phi(t)$ of a vector field A^μ at the background level (where t is the cosmic time). There the spatial vector component was treated as a perturbation on the FLRW background. In concrete dark energy models it was found that $\phi(t)$ grows toward a de Sitter attractor, whereas the vector perturbation decays after entering the vector sound horizon. Thus the analysis of Refs. [30, 31] is self-consistent, but it remains to see whether or not the spatial vector component $v(t)$ as large as $\phi(t)$ can modify the cosmological dynamics. If $v(t)$ is non-negligible relative to $\phi(t)$, we need to take into account the spatial shear $\sigma(t)$ in the metric as well. In the context of anisotropic inflation, for example, it is known that there are cases in which the ratio between the anisotropic and isotropic expansion rates stays nearly constant [34]. In generalized Proca theories, we would like to clarify the behavior of $v(t)$ and the anisotropic expansion rate $\Sigma(t) = \dot{\sigma}(t)$ during the cosmic expansion history from the radiation era to the late-time accelerated epoch.

In beyond-generalized Proca theories, it is important to study whether the Ostrogradski ghost appears or not on the anisotropic cosmological background. In this paper, we show the absence of additional ghostly DOF by explicitly computing the Hamiltonian in the presence of v and Σ . An interesting property of beyond-generalized Proca theories with quartic and quintic Lagrangians is that the only consistent solution free from a determinant singularity of the dynamical system corresponds to $v = 0$. In this case, the cosmological dynamics can be well described by the isotropic one studied in Ref. [32].

We organize our paper as follows. In Sec. II we derive the Hamiltonian and the full equations of motion in beyond-generalized Proca theories on the anisotropic cosmological background. In Sec. III we analytically estimate the evolution of v and Σ in the radiation/matter eras and in the de Sitter epoch. In Sec. IV we perform the numerical study to clarify the cosmological dynamics for both generalized and beyond-generalized Proca theories in detail, paying particular attention to the evolution of the dark energy equation of state w_{DE} as well as v and Σ . Sec. V is devoted to conclusions.

II. EQUATIONS OF MOTION ON THE ANISOTROPIC COSMOLOGICAL BACKGROUND

The beyond-generalized Proca theories [32], which encompass the second-order generalized Proca theories as a specific case [24, 27], are described by the action

$$S = \int d^4x \sqrt{-g} \left(\mathcal{L}_F + \mathcal{L}_M + \sum_{i=2}^6 \mathcal{L}_i + \mathcal{L}^N \right), \quad (2.1)$$

where g is the determinant of metric $g_{\mu\nu}$, \mathcal{L}_F is the standard Maxwell term given by

$$\mathcal{L}_F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (2.2)$$

with $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, and \mathcal{L}_M is the matter Lagrangian density (∇_μ is the covariant derivative operator).

The second-order generalized Proca theories, which break the U(1) gauge-invariance in the presence of a vector mass term m , correspond to the Lagrangian densities $\mathcal{L}_{2,3,4,5,6}$ in Eq. (2.1). They are given, respectively, by [24, 27]

$$\mathcal{L}_2 = G_2(X), \quad (2.3)$$

$$\mathcal{L}_3 = G_3(X) \nabla_\mu A^\mu, \quad (2.4)$$

$$\mathcal{L}_4 = G_4(X) R + G_{4,X}(X) [(\nabla_\mu A^\mu)^2 - \nabla_\rho A_\sigma \nabla^\sigma A^\rho] + \frac{1}{2} g_4(X) F_{\mu\nu} F^{\mu\nu}, \quad (2.5)$$

$$\begin{aligned} \mathcal{L}_5 = & G_5(X) G_{\mu\nu} \nabla^\mu A^\nu - \frac{1}{6} G_{5,X}(X) [(\nabla_\mu A^\mu)^3 - 3 \nabla_\mu A^\mu \nabla_\rho A_\sigma \nabla^\sigma A^\rho + 2 \nabla_\rho A_\sigma \nabla^\gamma A^\rho \nabla^\sigma A_\gamma] \\ & - g_5(X) \tilde{F}^{\alpha\mu} \tilde{F}^{\beta\gamma} \nabla_\mu \nabla_\alpha A_\beta, \end{aligned} \quad (2.6)$$

$$\mathcal{L}_6 = G_6(X) L^{\mu\nu\alpha\beta} \nabla_\mu A_\nu \nabla_\alpha A_\beta + \frac{1}{2} G_{6,X}(X) \tilde{F}^{\alpha\beta} \tilde{F}^{\mu\nu} \nabla_\alpha A_\mu \nabla_\beta A_\nu, \quad (2.7)$$

where $G_{2,3,4,5,6}$ and $g_{4,5}$ are functions of $X = -A_\mu A^\mu/2$ with the notation of partial derivatives as $G_{i,X} \equiv \partial G_i/\partial X$, R is the Ricci scalar, and $G_{\mu\nu}$ is the Einstein tensor. The original massive Proca Lagrangian corresponds to $G_2(X) = m^2 X$. The quantities $L^{\mu\nu\alpha\beta}$ and $\tilde{F}^{\mu\nu}$ are the double dual Riemann tensor and the dual strength tensor defined, respectively, by

$$L^{\mu\nu\alpha\beta} = \frac{1}{4} \mathcal{E}^{\mu\nu\rho\sigma} \mathcal{E}^{\alpha\beta\gamma\delta} R_{\rho\sigma\gamma\delta}, \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \mathcal{E}^{\mu\nu\alpha\beta} F_{\alpha\beta}, \quad (2.8)$$

where $\mathcal{E}^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor and $R_{\rho\delta\gamma\delta}$ is the Riemann tensor. The Maxwell term (2.2) and the last term of Eq. (2.5) can be absorbed into the Lagrangian density \mathcal{L}_2 by allowing the dependence of $\mathcal{L}_F = -F_{\mu\nu} F^{\mu\nu}/4$ as $G_2 = G_2(X, \mathcal{L}_F)$ [24, 26, 31]. We separate the \mathcal{L}_F dependence from \mathcal{L}_2 because this allows us to see the kinetic term of a spatial component v of the vector field explicitly in the Lagrangian. It is also possible to include the dependence of the term $Y = A^\mu A^\nu F_{\mu}{}^\alpha F_{\nu\alpha}$ in \mathcal{L}_2 [31]. On the anisotropic cosmological background studied in this paper, the quantity Y can be expressed in terms of X and \mathcal{L}_F as $Y = 4X\mathcal{L}_F$, so we do not take into account such dependence.

The Lagrangian density \mathcal{L}^N in Eq. (2.1) corresponds to the one beyond the domain of second-order generalized Proca theories [32]. This is given by the sum of four contributions

$$\mathcal{L}^N = \mathcal{L}_4^N + \mathcal{L}_5^N + \tilde{\mathcal{L}}_5^N + \mathcal{L}_6^N, \quad (2.9)$$

where

$$\mathcal{L}_4^N = f_4(X) \delta_{\alpha_1\alpha_2\alpha_3\gamma_4}^{\beta_1\beta_2\beta_3\gamma_4} A^{\alpha_1} A_{\beta_1} \nabla^{\alpha_2} A_{\beta_2} \nabla^{\alpha_3} A_{\beta_3}, \quad (2.10)$$

$$\mathcal{L}_5^N = f_5(X) \delta_{\alpha_1\alpha_2\alpha_3\alpha_4}^{\beta_1\beta_2\beta_3\beta_4} A^{\alpha_1} A_{\beta_1} \nabla^{\alpha_2} A_{\beta_2} \nabla^{\alpha_3} A_{\beta_3} \nabla^{\alpha_4} A_{\beta_4}, \quad (2.11)$$

$$\tilde{\mathcal{L}}_5^N = \tilde{f}_5(X) \delta_{\alpha_1\alpha_2\alpha_3\alpha_4}^{\beta_1\beta_2\beta_3\beta_4} A^{\alpha_1} A_{\beta_1} \nabla^{\alpha_2} A^{\alpha_3} \nabla_{\beta_2} A_{\beta_3} \nabla^{\alpha_4} A_{\beta_4}, \quad (2.12)$$

$$\mathcal{L}_6^N = f_6(X) \delta_{\alpha_1\alpha_2\alpha_3\alpha_4}^{\beta_1\beta_2\beta_3\beta_4} \nabla_{\beta_1} A_{\beta_2} \nabla^{\alpha_1} A^{\alpha_2} \nabla_{\beta_3} A^{\alpha_3} \nabla_{\beta_4} A^{\alpha_4}, \quad (2.13)$$

with $\delta_{\alpha_1\alpha_2\alpha_3\gamma_4}^{\beta_1\beta_2\beta_3\gamma_4} = \mathcal{E}_{\alpha_1\alpha_2\gamma_3\gamma_4}^{\beta_1\beta_2\gamma_3\gamma_4} \mathcal{E}^{\beta_1\beta_2\gamma_3\gamma_4}$, and the functions $f_{4,5,6}, \tilde{f}_5$ depend on X . Taking the limit $A^\mu \rightarrow \nabla^\mu \pi$, the Lagrangian densities \mathcal{L}_4^N and \mathcal{L}_5^N of the scalar field π are equivalent to those appearing in GLPV theories [14]. In GLPV theories, such terms do not give rise to an extra DOF associated with the Ostrogradski ghost. For the vector-field Lagrangian densities (2.10)-(2.13) it was shown in Ref. [32] that additional propagating DOFs to those appearing in second-order generalized Proca theories (one longitudinal mode and two transverse polarizations) do not arise on the maximally symmetric space-time and for linear cosmological perturbations on the flat FLRW background.

In Refs. [30, 31] the cosmology in generalized Proca theories was studied on the flat FLRW background under the assumption that the vector field A^μ has a time-dependent temporal component $\phi(t)$ alone. The spatial part of A^μ was treated as the perturbations on the FLRW background. In this paper, we would like to explicitly include the spatial component $v(t)$ of A^μ besides the temporal component $\phi(t)$ already present at the background level. For concreteness we consider the vector field A^i pointing to the x -direction. Since there is the rotational symmetry in the (y, z) plane, we take the line-element in the following form [34]

$$ds^2 = -N^2(t) dt^2 + e^{2\alpha(t)} \left[e^{-4\sigma(t)} dx^2 + e^{2\sigma(t)} (dy^2 + dz^2) \right], \quad (2.14)$$

where $N(t)$ is the lapse, $e^\alpha \equiv a$ is the isotropic scale factor, and σ characterizes the deviation from isotropy. We write the vector field in the form

$$A^\mu = \left(\frac{\phi(t)}{N(t)}, e^{-\alpha(t)+2\sigma(t)} v(t), 0, 0 \right), \quad (2.15)$$

in which case the term X is given by

$$X = \frac{1}{2} \phi^2(t) - \frac{1}{2} v^2(t). \quad (2.16)$$

Expanding the action (2.1) for the line element (2.14) and integrating the second derivatives $\ddot{\alpha}$ and $\ddot{\sigma}$ by parts, we obtain the action $S = \int d^4x L$ with the Lagrangian

$$L = \frac{e^{3\alpha}}{N^3} (\mathcal{F}_1 + \mathcal{F}_2 \dot{\sigma}^2 + \mathcal{F}_3 \dot{v}^2 + \mathcal{F}_4 \dot{v} v + \mathcal{F}_5 v^2 + N^4 \mathcal{L}_M), \quad (2.17)$$

where a dot represents the derivative with respect to t , and

$$\mathcal{F}_1 = N \left[N^3 G_2 + N^2 G_3 (\dot{\phi} + 3\dot{\alpha}\phi) + 6N\dot{\alpha}^2 (G_{4,X}\phi^2 - G_4) - G_{5,X}\dot{\alpha}^3 \phi^3 + 6\dot{\alpha}^2 \phi^4 (Nf_4 + f_5\dot{\alpha}\phi) \right], \quad (2.18)$$

$$\mathcal{F}_2 = N \left[6NG_4 - 6NG_{4,X}\phi^2 + G_{5,X}\phi^3 (3\dot{\alpha} + 2\dot{\sigma}) - 6Nf_4\phi^4 - 6f_5\phi^5 (3\dot{\alpha} + 2\dot{\sigma}) \right], \quad (2.19)$$

$$\mathcal{F}_3 = \frac{1}{2} (1 - 2g_4) N^2 - (\dot{\alpha} + \dot{\sigma}) \left[2Ng_5\phi - (\dot{\alpha} + \dot{\sigma}) \{ G_6 + (G_{6,X} + 2f_6)\phi^2 \} \right], \quad (2.20)$$

$$\begin{aligned} \mathcal{F}_4 = & \dot{\alpha} \left[2(G_{6,X} + 2f_6)\dot{\alpha}^2 \phi^2 + N^2 (1 + 4G_{4,X} + 4f_4\phi^2 - 2g_4) - N\dot{\alpha}\phi (G_{5,X} - 6f_5\phi^2 + 4g_5) \right] \\ & + 2G_6(\dot{\alpha} - 2\dot{\sigma})(\dot{\alpha} + \dot{\sigma})^2 - 4(G_{6,X} + 2f_6)\phi^2 \dot{\sigma}^3 + \phi [8Ng_5 - NG_{5,X} - 6(G_{6,X} + 2f_6)\dot{\alpha}\phi + 6Nf_5\phi^2] \dot{\sigma}^2 \\ & + 2N [N(2G_{4,X} - 1 + 2f_4\phi^2 + 2g_4) - \dot{\alpha}\phi (G_{5,X} - 6f_5\phi^2 - 2g_5)] \dot{\sigma}, \end{aligned} \quad (2.21)$$

$$\begin{aligned} \mathcal{F}_5 = & G_6(\dot{\alpha} - 2\dot{\sigma})^2 (\dot{\alpha} + \dot{\sigma})^2 + 4(G_{6,X} + 2f_6)\phi^2 \dot{\sigma}^4 - 4\phi [2Ng_5 - \phi (G_{6,X}\dot{\alpha} + 3Nf_5\phi + 2f_6\dot{\alpha})] \dot{\sigma}^3 \\ & - [3(G_{6,X} + 2f_6)\dot{\alpha}^2 \phi^2 - 2N^2 (1 + 3f_4\phi^2 - 2g_4) + 6Nf_5\phi^2 (\dot{\phi} - 3\dot{\alpha}\phi)] \dot{\sigma}^2 \\ & - 2[N\dot{\alpha} (N + 6f_5\phi^2 \dot{\phi} - 2Ng_4) + (G_{6,X} + 2f_6)\dot{\alpha}^3 \phi^2 + 2N^2 f_4 \phi \dot{\phi} - 3Ng_5 \dot{\alpha}^2 \phi] \dot{\sigma} \\ & + \frac{1}{2} \dot{\alpha} [N\dot{\alpha} \{ (1 - 2g_4)N - 12\phi^2 (Nf_4 + f_5\dot{\phi}) \} + 2(G_{6,X} + 2f_6)\dot{\alpha}^3 \phi^2 - 8N^2 f_4 \phi \dot{\phi} - 4N\dot{\alpha}^2 \phi (3f_5\phi^2 + g_5)]. \end{aligned} \quad (2.22)$$

In the isotropic case we have that $\sigma = 0$ and $v = 0$, so the Lagrangian (2.17) reduces to $L = e^{3\alpha}(\mathcal{F}_1 + N^4 \mathcal{L}_M)/N^3$. When the spatial anisotropy is present, the terms containing $\mathcal{F}_{2,3,4,5}$ in Eq. (2.17) contribute to the dynamics. On the anisotropic background we are studying here, the $\tilde{\mathcal{L}}_5^N$ term does not contribute to the dynamics at all.

For the matter sector, we consider a perfect fluid in terms of the k-essence description, i.e., $\mathcal{L}_M = P(Z)$, where $Z = -g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi / 2$ is the kinetic term of a scalar field χ [35]. Then, the matter Lagrangian $L_M = \sqrt{-g} \mathcal{L}_M$, which corresponds to the last term of Eq. (2.17), reads

$$L_M = Ne^{3\alpha} P(Z(N)), \quad (2.23)$$

where $Z(N) = \dot{\chi}^2 / (2N^2)$. Varying L_M with respect to χ and setting $N = 1$ at the end, we obtain the continuity equation

$$\dot{\rho}_M + 3\dot{\alpha}(\rho_M + P_M) = 0, \quad (2.24)$$

where ρ_M and P_M are given by

$$\rho_M = 2ZP_{,Z} - P, \quad P_M = P, \quad (2.25)$$

which correspond to the energy density and the pressure, respectively. Note that ρ_M and P_M appear in the background equations of motion derived by the variations of N and α , respectively.

The Lagrangian (2.17) contains the time-derivatives of $\alpha, \sigma, \phi, v, \chi$ up to first order, so the resulting equations of motion for these variables remain of second order. If we compute the determinant \mathcal{D} of the 6×6 Hessian matrix

$$H_L^{\mu\nu} = \frac{\partial^2 L}{\partial \dot{\mathcal{O}}_\mu \partial \dot{\mathcal{O}}_\nu}, \quad (2.26)$$

where $\mathcal{O} = (N(t), \alpha(t), \sigma(t), \phi(t), v(t), \chi(t))$, it follows that $\mathcal{D} = 0$ due to the absence of time derivatives of N in Eq. (2.17). This suggests the existence of a constraint that forbids the propagation of an additional ghostly DOF on that background. In fact, variation of (2.17) with respect to N leads to the constraint equation

$$\frac{\partial L}{\partial N} = 0. \quad (2.27)$$

Defining the conjugate momentum $\Pi^\mu = \partial L / \partial \dot{\mathcal{O}}_\mu$, the Hamiltonian of the system is given by $\mathcal{H} = \Pi^\mu \dot{\mathcal{O}}_\mu - L$. Introducing the isotropic expansion rate H and the anisotropic expansion rate Σ , as

$$H \equiv \frac{\dot{\alpha}}{N}, \quad \Sigma \equiv \frac{\dot{\sigma}}{N}, \quad (2.28)$$

the Hamiltonian reads

$$\mathcal{H} = e^{3\alpha} \left[N\rho_M - C_1(H + \Sigma)\dot{\phi} - C_2 \frac{\dot{v}^2}{N} - C_3 \dot{v} - NC_4 \right], \quad (2.29)$$

where

$$C_1 = 4\phi v^2 [f_4 + 3\phi(H + \Sigma)f_5] , \quad (2.30)$$

$$C_2 = -\frac{1}{2} + g_4 + 4\phi(H + \Sigma)g_5 - 3(H + \Sigma)^2 [G_6 + \phi^2(G_{6,X} + 2f_6)] , \quad (2.31)$$

$$C_3 = 2v [(H - 2\Sigma)C_2 - 2(H + \Sigma)(G_{4,X} + \phi^2 f_4) - \phi(H + \Sigma)^2(6\phi^2 f_5 - G_{5,X})] , \quad (2.32)$$

$$C_4 = G_2 + 6(H^2 - \Sigma^2) [G_4 - \phi^2 G_{4,X} - \phi^2(\phi^2 - v^2)f_4] + 2\phi^3(H - 2\Sigma)(H + \Sigma)^2 [G_{5,X} - 6(\phi^2 - v^2)f_5] \\ + v^2(H - 2\Sigma)^2 C_2 . \quad (2.33)$$

The Hamiltonian (2.29) is related to the quantity $\partial L / \partial N$, as

$$\frac{\partial L}{\partial N} = -\frac{\mathcal{H}}{N} , \quad (2.34)$$

so the constraint (2.27) translates to

$$\mathcal{H} = 0 . \quad (2.35)$$

This means that the above system is not plagued by the Ostrogradski instability associated with the Hamiltonian unbounded from below [33]. Since the coefficient C_1 depends on the two functions f_4 and f_5 , the derivative term $C_1(H + \Sigma)\dot{\phi}$ in Eq. (2.29) arises outside the domain of second-order generalized Proca theories. However, the Lagrangian densities \mathcal{L}_4^N and \mathcal{L}_5^N do not give rise to the appearance of an additional dangerous DOF related to the Ostrogradski ghost. Thus, the beyond-generalized theories remain healthy not only on the isotropic FLRW background [32] but also on the anisotropic cosmological background.

Varying the Lagrangian (2.17) with respect to N , α , σ , ϕ , v , respectively, and setting $N = 1$ at the end, we obtain the dynamical equations of motion

$$C_1(H + \Sigma)\dot{\phi} + C_2\dot{v}^2 + C_3\dot{v} + C_4 = \rho_M , \quad (2.36)$$

$$C_1\ddot{\phi} + (C_5\dot{v} + C_6)\ddot{v} + (C_7\dot{\phi} + C_8\dot{v}^2 + C_9\dot{v} + C_{10})\dot{H} + (C_7\dot{\phi} + C_8\dot{v}^2 + C_{11}\dot{v} + C_{12})\dot{\Sigma} \\ + C_{13}\dot{\phi}^2 + (C_{14}\dot{v}^2 + C_{15}\dot{v} + C_{16})\dot{\phi} + C_{17}\dot{v}^3 + C_{18}\dot{v}^2 + C_{19}\dot{v} + C_{20} = -3P_M , \quad (2.37)$$

$$\frac{d}{dt} \left[e^{3\alpha} \left(C_1\dot{\phi} + \frac{C_5}{2}\dot{v}^2 + C_{21}\dot{v} + C_{22} \right) \right] = 0 , \quad (2.38)$$

$$C_1(\dot{H} + \dot{\Sigma}) + D_1\dot{v}^2 + D_2\dot{v} + D_3 = 0 , \quad (2.39)$$

$$D_4\ddot{v} - (C_5\dot{v} + C_6)\dot{H} - (C_5\dot{v} + C_{21})\dot{\Sigma} + (2D_1\dot{v} + D_2)\dot{\phi} + D_5\dot{v}^2 + 3HD_4\dot{v} + D_6 = 0 , \quad (2.40)$$

where the coefficients $C_{5,\dots,22}$ and $D_{1,\dots,6}$ are given in Appendix A. The terms containing C_1 appear in the beyond-generalized Proca theories. In this case, the dynamical evolution of the spatial vector component v is different from that in the generalized Proca theories. In Sec. IV we shall discuss the cosmological dynamics in both generalized and beyond-generalized Proca theories in detail.

III. ANALYTIC SOLUTIONS TO THE SPATIAL ANISOTROPY

Let us analytically estimate the evolution of the anisotropic expansion rate Σ and the spatial vector component v . To recover the expansion history close to that of GR in the early cosmological epoch, we consider the function

$$G_4(X) = \frac{M_{\text{pl}}^2}{2} + \tilde{G}_4(X) , \quad (3.1)$$

where M_{pl} is the reduced Planck mass, and $\tilde{G}_4(X)$ is a function of X . We write Eqs. (2.36) and (2.37) in the following forms

$$3M_{\text{pl}}^2 H^2 = \rho_M + \rho_{\text{DE}} , \quad (3.2)$$

$$M_{\text{pl}}^2 (2\dot{H} + 3H^2) = -P_M - P_{\text{DE}} , \quad (3.3)$$

where ρ_{DE} and P_{DE} correspond to the energy density and the pressure of the “dark” component originating from the vector field, respectively. Introducing the density parameters

$$\Omega_M = \frac{\rho_M}{3M_{\text{pl}}^2 H^2}, \quad \Omega_{\text{DE}} = \frac{\rho_{\text{DE}}}{3M_{\text{pl}}^2 H^2}, \quad (3.4)$$

we obtain the constraint $\Omega_M + \Omega_{\text{DE}} = 1$ from Eq. (3.2). We also define the effective equation of state w_{eff} and the dark energy equation of state w_{DE} , as

$$w_{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2}, \quad w_{\text{DE}} = \frac{P_{\text{DE}}}{\rho_{\text{DE}}}. \quad (3.5)$$

For the matter sector, we take into account radiation (labelled by “ r ”) and non-relativistic matter (labelled by “ m ”), such that $\rho_M = \rho_r + \rho_m$ and $P_M = \rho_r/3$. Along the line of Refs. [30, 31], the existence of a late-time de Sitter solution with $w_{\text{DE}} = -1$ is assumed for the analytic estimation in Sec. III B. In Sec. IV we will present a concrete dark energy model with a de Sitter attractor. The background expansion history is given by the cosmological sequence of radiation-dominated ($w_{\text{eff}} = 1/3$) \rightarrow matter-dominated ($w_{\text{eff}} = 0$) \rightarrow de Sitter ($w_{\text{eff}} = -1$) epochs.

A. Radiation and matter eras

During the radiation and deep matter eras, the dark energy density parameter Ω_{DE} and the quantity $\epsilon_{P_{\text{DE}}} = P_{\text{DE}}/(3M_{\text{pl}}^2 H^2)$ are much smaller than the order of unity. Except for the term $3M_{\text{pl}}^2 H^2$, each term appearing on the l.h.s. of Eq. (2.36) can be assumed to be much smaller than $3M_{\text{pl}}^2 H^2$, say, $|C_3 \dot{v}|/(3M_{\text{pl}}^2 H^2) \ll 1$. Since the first three terms inside the parenthesis of Eq. (2.38) are similar to those appearing in Eq. (2.36) (apart from the difference divided by H), we express them in the form

$$C_1 \dot{\phi} + \frac{C_5}{2} \dot{v}^2 + C_{21} \dot{v} = \epsilon M_{\text{pl}}^2 H, \quad (3.6)$$

where ϵ is a dimensionless parameter. Under the condition $|\Sigma| \ll H$, the dominant contribution to the term C_{22} is given by

$$C_{22} \simeq -6M_{\text{pl}}^2 \Sigma - v^2 \left(4HC_2 - \frac{5}{2} H^2 C_5 \right). \quad (3.7)$$

Then, the equation of motion (2.38) for Σ reads

$$\frac{d}{dt} \left[a^3 \left\{ \Sigma - \frac{1}{6} \epsilon H + \frac{v^2}{12M_{\text{pl}}^2} (8HC_2 - 5H^2 C_5) \right\} \right] = 0. \quad (3.8)$$

As long as the dark energy density is suppressed relative to the background fluid density, the parameter ϵ is much smaller than 1 in the early cosmological epoch. In Eq. (2.36) the term $-v^2 H^2/2$ exists in the coefficient C_4 , so we require the condition $v^2 \ll M_{\text{pl}}^2$ in the early cosmological epoch. Provided that the conditions

$$|\Sigma| \gg \left| \frac{1}{6} \epsilon H \right|, \quad |\Sigma| \gg \frac{v^2}{12M_{\text{pl}}^2} |8HC_2 - 5H^2 C_5| \quad (3.9)$$

are satisfied, Σ evolves as

$$\Sigma \propto a^{-3}. \quad (3.10)$$

We define the ratio between Σ and H , as

$$r_\Sigma \equiv \frac{\Sigma}{H}. \quad (3.11)$$

When Σ decreases as Eq. (3.10), the evolution of r_Σ during the radiation and matter eras is given, respectively, by

$$r_\Sigma \propto a^{-1} \quad (\text{radiation era}), \quad (3.12)$$

$$r_\Sigma \propto a^{-3/2} \quad (\text{matter era}). \quad (3.13)$$

If v is not very small and the second condition of Eq. (3.9) does not hold, it happens that the anisotropy is sustained by v in such a way that Σ balances the term containing v^2 in Eq. (3.8). In Sec. IV we shall study the evolution of Σ in concrete dark energy models and show that, depending on model parameters and initial conditions, the ratio r_Σ can stay constant in the radiation-dominated epoch. This means that there are cases in which r_Σ does not necessarily decrease as Eq. (3.12).

We also estimate the evolution of v under the condition that the ratio $|r_\Sigma|$ is much smaller than 1. Neglecting the contributions of the terms Σ and $\dot{\Sigma}$ to Eq. (2.40), the equation of motion for v reads

$$\ddot{v} + \left(3H + \frac{\alpha_1}{q_V}\right) \dot{v} + \left(2H^2 + \dot{H} + \frac{\alpha_2}{q_V}\right) v \simeq 0, \quad (3.14)$$

where

$$q_V = 1 - 2g_4 - 4H\phi g_5 + 2H^2 (G_6 + \phi^2 G_{6,X} + 2\phi^2 f_6), \quad (3.15)$$

$$\begin{aligned} \alpha_1 = & 4\dot{H} [HG_6 + \phi\{(G_{6,X} + 2f_6)H\phi - g_5\}] \\ & + 2\dot{\phi} (\phi H [H\{3G_{6,X} + 4f_6 + (G_{6,XX} + 2f_{6,X})\phi^2\} - 2\phi g_{5,X}] - g_{4,X}\phi - 2Hg_5), \end{aligned} \quad (3.16)$$

$$\begin{aligned} \alpha_2 = & G_{2,X} + 6H^2 G_{4,X} + 4\dot{H}(G_{4,X} + H^2 G_6) + \dot{v}^2 [g_{4,X} - H\{HG_{6,X} + \phi[(G_{6,XX} + 2f_{6,X})H\phi - 2g_{5,X}]\}] + 3H\phi G_{3,X} \\ & + 2\dot{H}\phi [2\phi\{f_4 + H(2Hf_6 + HG_{6,X} + 3\phi f_5)\} - HG_{5,X} - 2Hg_5] + \phi H^2 [\phi\{6G_{4,XX} + 24f_4 + \phi[30Hf_5 - HG_{5,XX} \\ & + 6\phi(f_{4,X} + H\phi f_{5,X})]\} - 3HG_{5,X}] + H^2 v^2 (H^2 G_{6,X} - g_{4,X} + \phi[H\{(G_{6,XX} + 2f_{6,X})H\phi - 6f_{5,X}\phi^2 - 2g_{5,X} \\ & - 6\phi f_{4,X}\}] + \dot{\phi} G_{3,X} + H\dot{\phi}\{[16f_4 - 2g_{4,X} + 4G_{4,XX} + 8H^2 f_6 + 6H^2 G_{6,X} + \phi\{4\phi f_{4,X} + H[30f_5 - 4g_{5,X} \\ & - G_{5,XX} + 2(G_{6,XX} + 2f_{6,X})H\phi + 6\phi^2 f_{5,X}]\} - 2(2f_{4,X} + 3H\phi f_{5,X})v^2] - HG_{5,X} - 4Hg_5). \end{aligned} \quad (3.17)$$

Note that we expressed the term g_4 in α_2 by using q_V . It is worthy of mentioning that the quantity q_V is identical to the coefficient appearing in front of the kinetic term of vector perturbations on the isotropic background [31, 32]. To avoid the appearance of ghosts associated with vector perturbations, we require that $q_V > 0$. Under the conditions

$$\left|\frac{\alpha_1}{q_V}\right| \ll H, \quad \left|\frac{\alpha_2}{q_V}\right| \ll H^2, \quad (3.18)$$

Eq. (3.14) approximately reduces to

$$\ddot{v} + 3H\dot{v} + (2H^2 + \dot{H})v \simeq 0. \quad (3.19)$$

If the effective equation of state w_{eff} defined by Eq. (3.5) is constant, the solution to Eq. (3.19) reads

$$v = c_1 a^{-(1-3w_{\text{eff}})/2} + c_2 a^{-1}, \quad (3.20)$$

where c_1 and c_2 are integration constants. The evolution of v in the early cosmological epoch is given by

$$v \propto a^0 \quad (\text{for } w_{\text{eff}} = 1/3), \quad (3.21)$$

$$v \propto a^{-1/2} \quad (\text{for } w_{\text{eff}} = 0). \quad (3.22)$$

Hence v stays nearly constant during the radiation era, but it decreases in proportion to $t^{-1/3}$ during the matter era.

If the conditions (3.18) are violated, the analytic solution (3.20) loses its validity. In such cases, however, the large contribution from the spatial vector component to the background energy density can affect the successful cosmic expansion history. In Sec. IV we shall study the dynamics of v in concrete dark energy models.

B. De Sitter fixed point

In the absence of v and Σ , it was shown in Refs. [30, 31] that there exist isotropic de Sitter solutions with constant ϕ and H for the functions $G_{2,3,4,5}$ containing the power-law term in X . Here, we would like to find other de Sitter fixed points at which v, Σ as well as ϕ, H are non-zero constants. To discuss the stability of solutions, we shall keep the time derivative $\dot{\Sigma}$ in Eqs. (2.38) and the derivative terms \ddot{v}, \dot{v} in Eq. (2.40), while dealing with H and ϕ as constants. Setting $\dot{v} = 0, \ddot{v} = 0$ in Eq. (2.38), it follows that

$$q_1 \dot{\Sigma} + q_2 \Sigma \simeq 3q_3 H^2 v^2, \quad (3.23)$$

where

$$q_1 = 6G_4 - 6\phi^2 G_{4,X} + 3\phi^3 [(G_{5,X} - 6\phi^2 f_5)(H + 2\Sigma) - 2\phi f_4] + v^2 [2 - 4g_4 - 3G_6(H^2 - 4H\Sigma - 8\Sigma^2) + 3\phi\{ \phi[2f_4 - H(HG_{6,X} - 6\phi f_5 + 2Hf_6) + 4\Sigma(HG_{6,X} + 3\phi f_5 + 2Hf_6) + 8\Sigma^2(G_{6,X} + 2f_6)] - 8g_5\Sigma\}], \quad (3.24)$$

$$q_2 = 3Hq_1 - 3H\Sigma[3\phi^3 G_{5,X} - 18\phi^3(\phi^2 - v^2)f_5 + 2v^2\{[G_6 + (G_{6,X} + 2f_6)\phi^2](3H + 8\Sigma) - 6\phi g_5\}], \quad (3.25)$$

$$q_3 = 1 - 2g_4 - 3H\phi g_5 + H^2(G_6 + \phi^2 G_{6,X} + 2\phi^2 f_6). \quad (3.26)$$

Setting $\dot{\Sigma} = 0$ in Eq. (2.40), keeping the terms linear in \dot{v} , \ddot{v} , and using Eq. (2.39), the equation of motion for v reads

$$D_4(\ddot{v} + 3H\dot{v}) + \frac{2(H + \Sigma)}{\phi}(H\phi q_V + q_4 - q_5 v^2)v \simeq 0, \quad (3.27)$$

where q_V is of the same form as Eq. (3.15), and

$$q_4 = \Sigma[18\phi^4(H + \Sigma)f_5 - 2\phi\{1 - 2g_4 - 3G_{4,X} + \Sigma(3H + 2\Sigma)G_6\} - \phi^3\{2\Sigma(3H + 2\Sigma)(G_{6,X} + 2f_6) - 6f_4\} - \phi^2\{3(H + \Sigma)G_{5,X} - 4(H + 2\Sigma)g_5\}],$$

$$q_5 = H^2(H\phi G_{6,X} + 2H\phi f_6 - g_5) + \Sigma[18(H + \Sigma)\phi^2 f_5 + 4(H - \Sigma)g_5 + 6\phi f_4 - \phi(G_{6,X} + 2f_6)(3H^2 - 4\Sigma^2)]. \quad (3.28)$$

On using Eqs. (3.23) and (3.27), we search for fixed points with constant values of v and Σ for $q_2 \neq 0$. One of them is the isotropic point characterized by

$$v = 0, \quad \Sigma = 0. \quad (3.29)$$

The other fixed points, which exist for $q_5 \neq 0$, obey the relations

$$v^2 = \frac{H\phi q_V + q_4}{q_5}, \quad (3.30)$$

$$\Sigma = \frac{3q_3 H^2 (H\phi q_V + q_4)}{q_2 q_5}. \quad (3.31)$$

Not only the quantities q_2, q_4, q_5 contain Σ , but also the quantities q_2, q_3, q_4, q_5, q_V depend on v through the dependence of functions $G_{4,5,6}, f_{4,5,6}, g_{4,5}$ with respect to $X = \phi^2/2 - v^2/2$. This means that, unless the functional forms of G_4 etc are specified, we cannot explicitly solve Eqs. (3.30) and (3.31) for v and Σ .

We are looking for the fixed points relevant to the late-time cosmic acceleration with a non-vanishing anisotropic expansion rate Σ . We assume that Σ is much smaller than H on the de Sitter solution, i.e.,

$$|r_\Sigma| \ll 1. \quad (3.32)$$

Moreover, we focus on the case in which the condition

$$v^2 \ll \phi^2 \quad (3.33)$$

is satisfied on the de Sitter solution. As we will study the dynamics of the vector field for concrete dark energy models in Sec. IV, the temporal component ϕ tends to grow toward the de Sitter fixed point in cosmologically viable cases. Meanwhile, as estimated by Eq. (3.22), the spatial component v typically decreases during the matter-dominated epoch. Hence, even if v is of the same order as ϕ in the radiation era, the condition (3.33) usually holds around the onset of cosmic acceleration. In what follows, we employ the approximation that the functions $G_{4,5,6}, f_{4,5,6}, g_{4,5}$ do not depend on v by dealing with the kinetic term as $X \simeq \phi^2/2$.

Under the approximation (3.32) we ignore the terms containing Σ in Eq. (3.25), in which case $q_2 \simeq 3Hq_1$. Moreover we also neglect the Σ -dependent terms in Eq. (3.27), in which case $D_4 \simeq q_V$. Then Eqs. (3.23) and (3.27) reduce, respectively, to

$$q_1(\dot{\Sigma} + 3H\Sigma) \simeq 3q_3 H^2 v^2, \quad (3.34)$$

$$q_V(\ddot{v} + 3H\dot{v} + 2H^2 v) \simeq \frac{2H^3}{\phi} \mathcal{A}_V v^3, \quad (3.35)$$

where

$$\mathcal{A}_V \equiv H\phi G_{6,X} + 2H\phi f_6 - g_5. \quad (3.36)$$

Let us first consider the theories characterized by

$$g_5 = 0, \quad G_6 = 0, \quad f_6 = 0, \quad (3.37)$$

in which case $\mathcal{A}_V = 0$ with $q_V = 1 - 2g_4$. Provided that $g_4 \neq 1/2$, Eq. (3.35) reduces to

$$\ddot{v} + 3H\dot{v} + 2H^2v = 0, \quad (3.38)$$

whose solution is given by

$$v = c_3 a^{-1} + c_4 a^{-2}, \quad (3.39)$$

where c_3 and c_4 are integration constants. Hence the spatial component v exponentially decreases toward 0. Since the r.h.s. of Eq. (3.34) approaches 0, the equation for Σ reduces to $\dot{\Sigma} + 3H\Sigma \simeq 0$. This means that the anisotropic expansion rate decays as $\Sigma \propto a^{-3}$. If $g_4 = 1/2$ then we have $q_3 = 1 - 2g_4 = 0$, so the r.h.s. of Eq. (3.34) vanishes. These discussions show that, for the theories given by the functions (3.37), both v and Σ decrease toward the isotropic fixed point (3.29).

We proceed to the theories in which the terms g_5, G_6, f_6 are present, such that

$$\mathcal{A}_V \neq 0. \quad (3.40)$$

In this case, besides the isotropic point (3.29), there exist other fixed points satisfying Eqs. (3.30) and (3.31). Ignoring the v dependence in the functions $G_{4,5,6}, f_{4,5,6}, g_{4,5}$ under the condition (3.33), the latter fixed points correspond to $v = v_c$ and $\Sigma = \Sigma_c$, where

$$v_c = \pm \sqrt{\frac{\phi q_V}{H \mathcal{A}_V}}, \quad \Sigma_c = \frac{q_3 q_V \phi}{\tilde{q}_1 \mathcal{A}_V}, \quad (3.41)$$

with

$$\tilde{q}_1 = 6G_4 - 6\phi^2 G_{4,X} + 3\phi^3 [(G_{5,X} - 6\phi^2 f_5)H - 2\phi f_4] + v_c^2 [2 - 4g_4 - 3H^2 G_6 + 3\phi^2 \{2f_4 - H(HG_{6,X} - 6\phi f_5 + 2Hf_6)\}]. \quad (3.42)$$

Existence of the fixed point (3.41) requires that $\phi q_V / (H \mathcal{A}_V) > 0$. Considering a linear perturbation δv around $v = v_c$ in Eq. (3.35), it follows that

$$\ddot{\delta v} + 3H\dot{\delta v} - 4H^2\delta v = 0, \quad (3.43)$$

whose solution is given by

$$\delta v = \tilde{c}_3 a + \tilde{c}_4 a^{-4}, \quad (3.44)$$

where \tilde{c}_3 and \tilde{c}_4 are integration constants. Since δv exponentially grows in time, the fixed point (3.41) is not stable. On the other hand, the perturbation around the fixed point $v = 0$ obeys the differential equation $\ddot{\delta v} + 3H\dot{\delta v} + 2H^2\delta v = 0$, so δv exponentially decreases toward 0. Then, the solutions finally approach the fixed point $v = 0$ rather than $v = v_c$. On using Eq. (3.34), Σ also decreases as $\propto a^{-3}$ toward the isotropic point $\Sigma = 0$. The above discussion shows that, under the conditions (3.32) and (3.33), the anisotropic hair with a non-vanishing Σ does not survive in general on the de Sitter background. This is consistent with the Wald's cosmic conjecture [36].

IV. ANISOTROPIC COSMOLOGICAL DYNAMICS IN CONCRETE DARK ENERGY MODELS

We study the anisotropic cosmological dynamics for a class of dark energy models in the framework of massive vector theories. We consider the functions $G_{2,3,4,5}$ containing the power-law dependence of X , such that

$$G_2(X) = b_2 X^{p_2}, \quad G_3(X) = b_3 X^{p_3}, \quad G_4(X) = \frac{M_{\text{pl}}^2}{2} + b_4 X^{p_4}, \quad G_5(X) = b_5 X^{p_5}, \quad (4.1)$$

where M_{pl} is the reduced Planck mass, $b_{2,3,4,5}$ and $p_{2,3,4,5}$ are constants. In the isotropic context ($v = 0 = \Sigma$), the simple solution $\phi^p \propto H^{-1}$ can be realized for the powers [30, 31]

$$p_3 = \frac{1}{2}(p + 2p_2 - 1), \quad p_4 = p + p_2, \quad p_5 = \frac{1}{2}(3p + 2p_2 - 1), \quad (4.2)$$

which accommodate the vector Galileon [24] as a specific case ($p_2 = 1$ and $p = 1$). Provided that $p > 0$, the temporal component ϕ grows with the decrease of H . Finally, the solutions approach the de Sitter fixed point characterized by constant ϕ and H . On the FLRW background the $G_6(X)$ term does not contribute to the dynamics [31].

In the isotropic setting, the dark energy density $\rho_{\text{DE}}^{(\text{iso})}$ and the pressure $P_{\text{DE}}^{(\text{iso})}$ originate from the temporal vector component ϕ . In this case, the dark energy equation of state $w_{\text{DE}}^{(\text{iso})} = P_{\text{DE}}^{(\text{iso})}/\rho_{\text{DE}}^{(\text{iso})}$ is analytically known as [30]

$$w_{\text{DE}}^{(\text{iso})} = -\frac{3(1+s) + s\Omega_r}{3(1+s\Omega_{\text{DE}})}, \quad (4.3)$$

where $s \equiv p_2/p$, and $\Omega_r = \rho_r/(3M_{\text{pl}}^2 H^2)$ is the radiation density parameter. The evolution of $w_{\text{DE}}^{(\text{iso})}$ is given by $w_{\text{DE}}^{(\text{iso})} \simeq -1 - 4s/3$ in the radiation-dominated epoch ($\Omega_r \simeq 1, \Omega_{\text{DE}} \simeq 0$), $w_{\text{DE}}^{(\text{iso})} \simeq -1 - s$ in the matter-dominated epoch ($\Omega_r \simeq 0, \Omega_{\text{DE}} \simeq 0$), and $w_{\text{DE}}^{(\text{iso})} \simeq -1$ during the dark energy dominance ($\Omega_r \simeq 0, \Omega_{\text{DE}} \simeq 1$). For $s > 0$ the phantom dark energy equation of state can be realized during the radiation and matter eras, but the parameter s is constrained to be $s \leq 0.36$ at 95 % confidence level for the compatibility with observations [37].

In the anisotropic setting, there are contributions to the background equations of motion (3.2) and (3.3) originating from v and Σ besides $\rho_{\text{DE}}^{(\text{iso})}$ and $P_{\text{DE}}^{(\text{iso})}$. As estimated by Eq. (3.21), let us consider the case in which v stays nearly constant during the radiation era with $|r_\Sigma| \ll 1$. Then, the dominant contribution to the dark energy density ρ_{DE} arising from v is the term $C_2 v^2 H^2$ in Eq. (2.33). Moreover, the contribution $3M_{\text{pl}}^2 \Sigma^2$ in C_4 cannot be necessarily neglected relative to $\rho_{\text{DE}}^{(\text{iso})}$ even for $|r_\Sigma| \ll 1$. Similarly, the terms $3C_2 v^2 H^2$ and $9M_{\text{pl}}^2 \Sigma^2$ in C_{20} and the v^2 -dependent term in C_{10} give rise to the dominant contribution to the pressure P_{DE} . Then, during the radiation era, we can estimate ρ_{DE} and P_{DE} , as

$$\rho_{\text{DE}} \simeq \rho_{\text{DE}}^{(\text{iso})} - C_2 v^2 H^2 + 3M_{\text{pl}}^2 \Sigma^2, \quad (4.4)$$

$$P_{\text{DE}} \simeq P_{\text{DE}}^{(\text{iso})} + v^2 H^2 \left(C_2 - \frac{2}{3} C_{10} \right) + 3M_{\text{pl}}^2 \Sigma^2, \quad (4.5)$$

where

$$C_2 = -\frac{1}{2} q_V + 2H\phi g_5 - 2H^2 (G_6 + \phi^2 G_{6,X} + 2\phi^2 f_6), \quad (4.6)$$

$$C_{10} = -q_V + 8H\phi g_5 - 10H^2 (G_6 + \phi^2 G_{6,X} + 2\phi^2 f_6) + 12\phi^2 (f_4 + 3H\phi f_5). \quad (4.7)$$

As we go back to the past, the isotropic contributions to Eqs. (4.4) and (4.5) get smaller [30], whereas the terms containing v and Σ tend to be larger. Provided that the two conditions

$$|C_2| \frac{v^2}{M_{\text{pl}}^2} \gg r_\Sigma^2, \quad |C_2| v^2 H^2 \gg \rho_{\text{DE}}^{(\text{iso})} \quad (4.8)$$

are satisfied with $|P_{\text{DE}}^{(\text{iso})}|$ and $|C_{10}|$ same order as $\rho_{\text{DE}}^{(\text{iso})}$ and $|C_2|$, respectively, the dark energy equation of state $w_{\text{DE}} = P_{\text{DE}}/\rho_{\text{DE}}$ is approximately given by

$$w_{\text{DE}} \simeq -1 + \frac{2C_{10}}{3C_2}. \quad (4.9)$$

If the terms containing $g_5, G_6, f_{4,5,6}$ provide the sub-dominant contributions to C_2 and C_{10} , we have that $C_{10} \simeq 2C_2$ and hence Eq. (4.9) reduces to $w_{\text{DE}} \simeq 1/3$. In this case, the spatial vector component v works as a dark radiation.

On the other hand, if the anisotropic expansion rate is large such that the conditions

$$r_\Sigma^2 \gg |C_2| \frac{v^2}{M_{\text{pl}}^2}, \quad r_\Sigma^2 \gg \frac{\rho_{\text{DE}}^{(\text{iso})}}{3M_{\text{pl}}^2 H^2} \quad (4.10)$$

are satisfied, it follows that

$$w_{\text{DE}} \simeq 1. \quad (4.11)$$

The above estimations are valid during the radiation era in which the isotropic contributions to Eqs. (4.4) and (4.5) are smaller than the terms containing v and Σ , but after the temporal vector component ϕ dominates the dynamics, the evolution of w_{DE} is described by $w_{\text{DE}}^{(\text{iso})}$ in Eq. (4.3).

In the following we shall study the cosmological dynamics in the three models: (A) $G_{2,3,4,5} \neq 0, G_6 = 0, g_4 \neq 0, g_5 = 0, f_{4,5,6} = 0$, (B) $G_{2,3,4,5} \neq 0, G_6 \neq 0, g_4 = 0, g_5 \neq 0, f_{4,5,6} = 0$, and (C) $f_{4,5,6} \neq 0$ with all the other functions $G_{2,3,4,5,6}, g_{4,5}$ non-vanishing. For $G_2(X)$, we take the functional form

$$G_2(X) = -m^2 X, \quad (4.12)$$

where m is a constant having a dimension of mass. Since $p_2 = 1$ in this case, the parameter s in Eq. (4.3) is equivalent to $s = 1/p$. The minus sign of $G_2(X)$ is chosen to avoid the tensor ghost and instability at the de Sitter fixed point [30]. Since the effective mass squared of the vector field is $2H^2$ on the de Sitter solution [30, 31], the tachyonic instability is absent even for the choice (4.12). We shall consider the case in which the bare mass m is of the order of the present Hubble parameter H_0 with the temporal component $\phi_0 \approx \mathcal{O}(M_{\text{pl}})$, such that the quantity defined by

$$\xi \equiv \frac{H}{m} \left(\frac{\phi}{M_{\text{pl}}} \right)^p \quad (4.13)$$

is of the order of unity. If ϕ dominates over v during the cosmological evolution, the quantity ξ stays nearly constant. We also introduce the following dimensionless constants

$$a_3 = \frac{M_{\text{pl}}^p b_3}{2^{p_3} m}, \quad a_4 = \frac{M_{\text{pl}}^{2p} b_4}{2^{p_4}}, \quad a_5 = \frac{M_{\text{pl}}^{3p} m b_5}{2^{p_5}}, \quad (4.14)$$

for the numerical purpose.

The structure of Eqs. (2.36)-(2.40) is different depending on whether the Lagrangian densities (2.10) and (2.11) are present or not. If $f_4 = f_5 = 0$, then the terms containing C_1 in Eqs. (2.36)-(2.39) vanish identically. In such cases, we take the time derivative of Eq. (2.39) and solve for \ddot{v} , $\dot{\phi}$, \dot{H} , and $\dot{\Sigma}$ by using Eqs. (2.37), (2.38), and (2.40). Then the dynamical equations are expressed in the autonomous form

$$Z\mathbf{x} = \mathbf{y}, \quad (4.15)$$

where $\mathbf{x} = {}^t(\ddot{v}, \dot{\phi}, \dot{H}, \dot{\Sigma})$, Z and \mathbf{y} are the 4×4 and 1×4 matrices, respectively, containing the dependence of $\dot{v}, v, \phi, H, \Sigma$. Provided that the determinant of Z does not vanish, we can solve Eq. (4.15) for \mathbf{x} , as $\mathbf{x} = Z^{-1}\mathbf{y}$. As usual, the Friedmann equation (2.36) can be used as a constraint equation.

In beyond-generalized Proca theories ($f_{4,5} \neq 0$), there are extra derivative terms related to the non-vanishing coefficient C_1 . Then the structure of the dynamical system is different from the one discussed above. In Sec. IV C we shall separately study such cases.

$$\mathbf{A.} \quad G_{2,3,4,5} \neq 0, G_6 = 0, g_4 \neq 0, g_5 = 0, f_{4,5,6} = 0$$

In this case the quantity q_V is equivalent to $1 - 2g_4$. Let us consider the model with constant g_4 with $q_V > 0$, i.e., $g_4 < 1/2$. In the radiation-dominated epoch, the dominant contribution to the Friedmann equation (2.36) originating from v corresponds to the energy density $\rho_{g_4} = -C_2 H^2 v^2 = q_V H^2 v^2 / 2$. The density parameter associated with ρ_{g_4} is given by

$$\Omega_{g_4} \equiv \frac{\rho_{g_4}}{3M_{\text{pl}}^2 H^2} = \frac{q_V}{6} \frac{v^2}{M_{\text{pl}}^2}. \quad (4.16)$$

Provided that $v^2 \ll M_{\text{pl}}^2$, Ω_{g_4} is much smaller than 1 for $q_V \lesssim 1$. In the following, we shall consider the case in which the condition $\Omega_{g_4} \ll 1$ is satisfied in the deep radiation era with $v^2 \lesssim \phi^2$.

For the model under consideration we have $\alpha_1 = 0$ in Eq. (3.14), whereas the quantity α_2 reduces to

$$\begin{aligned} \alpha_2 = & -m^2 + (\dot{\phi} + 3H\phi)G_{3,X} + 2(3H^2 + 2\dot{H})G_{4,X} + 2H\phi(2\dot{\phi} + 3H\phi)G_{4,XX} \\ & - H(H\dot{\phi} + 2\dot{H}\phi + 3H^2\phi)G_{5,X} - H^2\phi^2(\dot{\phi} + H\phi)G_{5,XX}. \end{aligned} \quad (4.17)$$

As long as the condition $|\alpha_2/q_V| \ll H^2$ is satisfied in the early cosmological epoch, the solution to Eq. (3.14) is given by Eq. (3.21) in the radiation era and by Eq. (3.22) in the matter era.

If v is nearly constant during the radiation domination, the terms containing v^2 in Eq. (3.8) affects the evolution of Σ . Neglecting the contribution $\epsilon H/6$ relative to Σ and using the fact that $C_2 = -q_V/2$ and $C_5 = 0$, the solution to Eq. (3.8) is given by

$$r_\Sigma \simeq \frac{q_V v^2}{3M_{\text{pl}}^2} + \frac{\mathcal{B}}{a^3 H}, \quad (4.18)$$

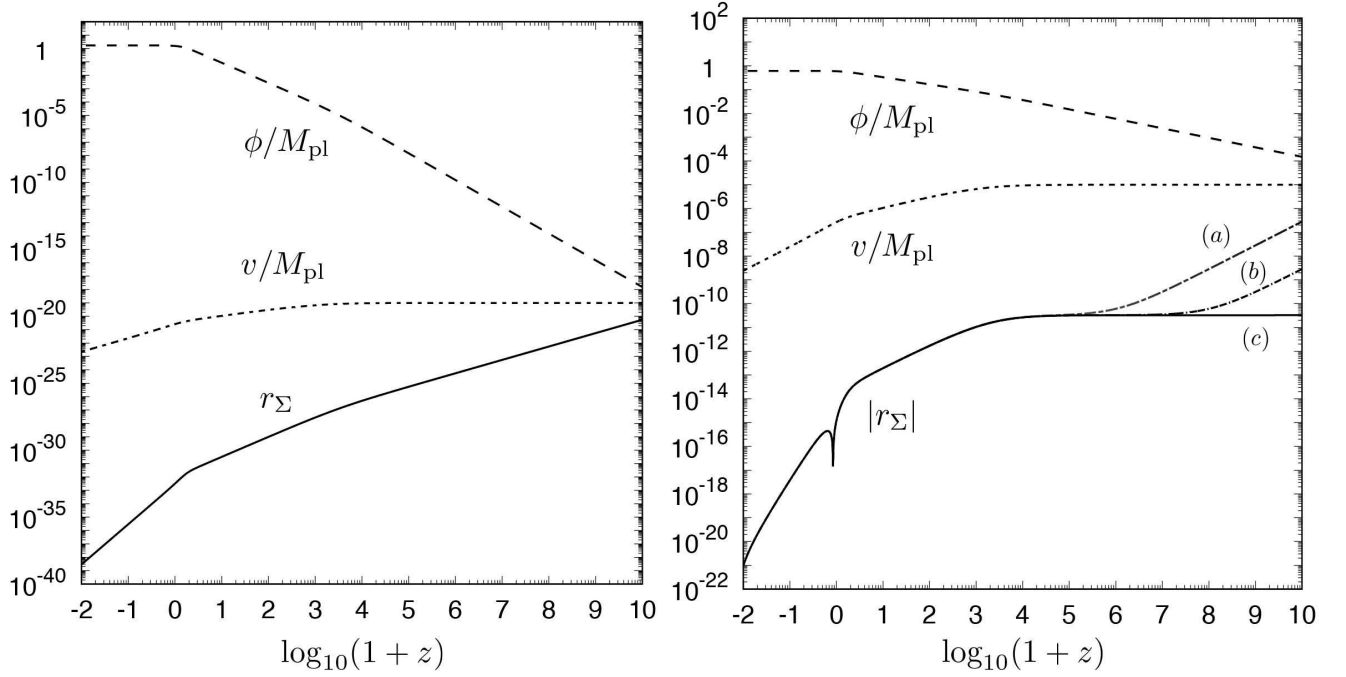


FIG. 1: Evolution of ϕ/M_{pl} , v/M_{pl} , and r_{Σ} in the model (A) with the parameters $a_4 = 0.01$, $a_5 = 0.05$, and $g_4 = 0.01$ for $p = 1$ (left) and $p = 5$ (right). We choose the initial conditions as $\phi/M_{\text{pl}} = 5.0 \times 10^{-19}$, $v/M_{\text{pl}} = 1.0 \times 10^{-19}$, $\dot{v} = 0$, $m\phi/(\sqrt{6}M_{\text{pl}}H) = 1.0 \times 10^{-37}$, $r_{\Sigma} = 5.0 \times 10^{-20}$ and $1 - \Omega_r = 1.7 \times 10^{-7}$ at the redshift $z = 1.8 \times 10^{10}$ (left), and $\phi/M_{\text{pl}} = 9.0 \times 10^{-5}$, $v/M_{\text{pl}} = 1.0 \times 10^{-5}$, $\dot{v} = 0$, $m\phi/(\sqrt{6}M_{\text{pl}}H) = 1.0 \times 10^{-23}$, and $1 - \Omega_r = 9.0 \times 10^{-8}$ at $z = 3.5 \times 10^{10}$ (right). The cases (a), (b), (c) in the right panel correspond to the three different initial conditions: (a) $r_{\Sigma} = 10^{-6}$, (b) $r_{\Sigma} = 10^{-8}$, and (c) $r_{\Sigma} = 3.3 \times 10^{-11}$ at $z = 3.5 \times 10^{10}$. The parameter a_3 is known from Eq. (2.39). The present epoch ($z = 0$) is identified by $\Omega_{\text{DE}} = 0.68$.

where \mathcal{B} is an integration constant. The first term on the r.h.s. of Eq. (4.18) stays constant, whereas the second term decreases as Eq. (3.12) in the radiation domination. Unless the first contribution is extremely smaller than the second one around the beginning of the radiation era, the anisotropic expansion rate should approach the value $r_{\Sigma} \simeq q_V v^2 / (3M_{\text{pl}}^2)$.

In the left panel of Fig. 1 we show an example for the evolution of v , ϕ , and r_{Σ} in the case of vector Galileons ($p = 1$) with $g_4 = 0.01$. In this simulation we confirmed that the condition $|\alpha_2/q_V| \ll H^2$ is well satisfied before the onset of cosmic acceleration, so the analytic solution (3.20) to v should be trustable. In fact v stays nearly constant during the radiation era, which is followed by the decrease of v after the matter dominance. In the left panel of Fig. 1 the temporal component ϕ grows as $\phi \propto H^{-1}$ toward the de Sitter attractor characterized by constant ϕ , whose property is similar to the isotropic case studied in Ref. [30].

For $p = 1$, the numerical simulation shows that Σ decreases as $\propto a^{-3}$ during the radiation and matter eras. This reflects the fact that the initial value of v , which is at most of the same order as ϕ , is very much smaller than M_{pl} ($v = 10^{-19}M_{\text{pl}}$ in the left panel of Fig. 1), so the system enters the matter-dominated epoch before r_{Σ} approaches the value $q_V v^2 / (3M_{\text{pl}}^2)$ in the radiation era. After the matter dominance, the solutions finally approach the isotropic fixed point characterized by $\Sigma = 0$ and $v = 0$. Since the model under consideration belongs to the class (3.37), there is no anisotropic de Sitter fixed point satisfying the conditions (3.32) and (3.33).

In the right panel of Fig. 1 we plot the evolution of v , ϕ , r_{Σ} for $p = 5$ and $g_4 = 0.01$ with three different initial values of r_{Σ} . The evolution of ϕ is similar to that in the isotropic case ($\phi \propto H^{-1/p}$), so the variation of ϕ tends to be milder for larger p (> 0). For $p = 5$, it is then possible to choose larger initial values of ϕ and v relative to those for $p = 1$. In the right panel of Fig. 1 the field v stays nearly constant ($v \simeq 10^{-5}M_{\text{pl}}$) during the radiation era, so the first term on the r.h.s. of Eq. (4.18) can be estimated as $q_V v^2 / (3M_{\text{pl}}^2) \simeq 3 \times 10^{-11}$. We choose several different initial values of r_{Σ} and find that the solutions temporally approach the value $r_{\Sigma} = q_V v^2 / (3M_{\text{pl}}^2)$ in the radiation era, see the curves (a), (b), (c) in Fig. 1. This shows that, for the models allowing large initial values of v (i.e., for greater p), the solution $r_{\Sigma} = q_V v^2 / (3M_{\text{pl}}^2)$ corresponds to the temporal attractor in the radiation-dominated epoch. After the onset of the matter domination, r_{Σ} starts to decrease with the decrease of v .

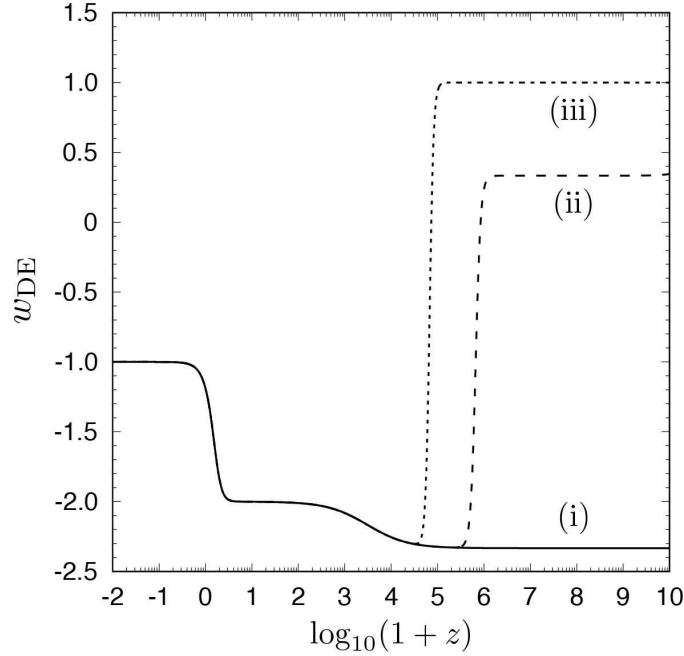


FIG. 2: Evolution of w_{DE} in the model (A) for $p = 1$ with $a_4 = 0.01$, $a_5 = 0.05$, and $g_4 = 0.01$. We choose the three different initial conditions: (i) $v = 0$ and $r_\Sigma = 0$, (ii) $v/M_{\text{Pl}} = 1.0 \times 10^{-19}$ and $r_\Sigma = 5.0 \times 10^{-20}$, and (iii) $v/M_{\text{Pl}} = 1.0 \times 10^{-19}$ and $r_\Sigma = 1.0 \times 10^{-10}$ at $z = 1.8 \times 10^{10}$. The initial conditions of other variables are the same as those used in the left panel of Fig. 1.

As for the early evolution of w_{DE} , we should notice that the quantities (4.6) and (4.7) reduce, respectively, to $\mathcal{C}_2 = -q_V/2$ and $\mathcal{C}_{10} = -q_V = 2\mathcal{C}_2$. If the conditions (4.8) are satisfied during the radiation era, the dark energy equation of state (4.9) reduces to $w_{\text{DE}} \simeq 1/3$. On the other hand, if the anisotropic expansion rate is initially large to satisfy the condition (4.10), we have that $w_{\text{DE}} \simeq 1$.

In Fig. 2 we plot the evolution of w_{DE} for $p = 1$ with three different initial values of v and Σ at the redshift $z = 10^{10}$. In the case (i) we have chosen the isotropic value $v = 0 = \Sigma$, so w_{DE} evolves according to Eq. (4.3), i.e., $-7/3$ (radiation era) $\rightarrow -2$ (matter era) $\rightarrow -1$ (de Sitter epoch). In the case (ii) the initial conditions are the same as those used in the left panel of Fig. 1. In this case the conditions (4.8) are satisfied in the deep radiation era, so that $w_{\text{DE}} \simeq 1/3$. After the contribution of ϕ to w_{DE} dominates over that of v , the evolution of w_{DE} is described by $w_{\text{DE}}^{(\text{iso})}$. In the numerical simulation of the case (ii), the approach of w_{DE} to $w_{\text{DE}}^{(\text{iso})}$ occurs around the redshift $z = 10^6$. In the case (iii) of Fig. 2, Σ is initially large to fulfill the conditions (4.10), so w_{DE} is close to 1. In this case Σ decreases in proportion to a^{-3} , so the solutions finally enter the regime in which $w_{\text{DE}} \simeq w_{\text{DE}}^{(\text{iso})}$ for $z \lesssim 10^5$. Realization of the case (iii) requires that Σ is of the order of $r_\Sigma \gtrsim 10^{-15}$ at $z = 10^{10}$.

For $p = 1$ the deviation of w_{DE} from -1 is significant during the matter era, so it is difficult for vector Galileons to be compatible with observations [37]. However, this situation is different for larger p (i.e., for smaller $s = 1/p$). In Fig. 3 we plot the evolution of w_{DE} for $p = 1, 3, 5$ with the initial values of v and Σ obeying the conditions (4.8) at $z = 10^{10}$. Hence w_{DE} starts to evolve from the value close to $1/3$, which is followed by the approach to the isotropic value $w_{\text{DE}}^{(\text{iso})} = -1 - 1/p$ in the matter era. For larger p the evolution of ϕ during the radiation era is milder (as seen in Fig. 1), so the difference between v and ϕ becomes less significant with the passage of time for the initial values of v same order as ϕ . Then, for larger p , the approach of w_{DE} to $w_{\text{DE}}^{(\text{iso})}$ tends to occur at the later cosmological epoch, but as long as $p = \mathcal{O}(1)$, the transition redshift is much larger than 1.

In the regime where the ratio r_Σ is close to $q_V v^2 / (3M_{\text{Pl}}^2)$, the last terms on the r.h.s. of Eqs. (4.4) and (4.5) are about v^2 / M_{Pl}^2 ($\ll 1$) times as small as the second terms, so the first condition of Eq. (4.8) is satisfied. During the radiation era in which the second condition of Eq. (4.8) is fulfilled as well, the spatial vector component behaves as a dark radiation with $w_{\text{DE}} \simeq 1/3$. In the case (iii) of Fig. 3, we can confirm that $w_{\text{DE}} \simeq 1/3$ by the time at which r_Σ starts to decrease from the value $q_V v^2 / (3M_{\text{Pl}}^2)$ around $z \approx 10^4$ (see the right panel of Fig. 1). Note that, in the gauge-quintessence scenario studied in Ref. [38], non-Abelian gauge fields also track the radiation during the radiation and matter eras. Here, the difference from Ref. [38] is that such a tracking behavior ends at high redshifts ($z \gg 1$).

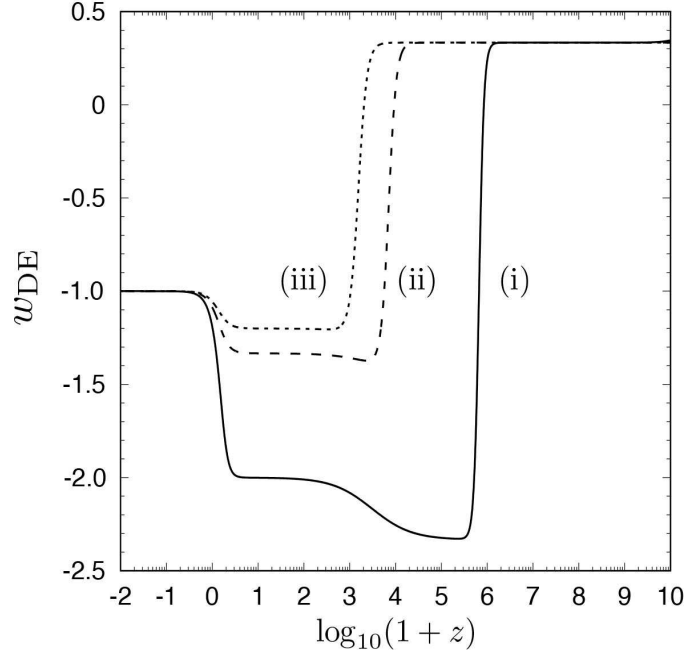


FIG. 3: Evolution of w_{DE} in the model (A) with $a_4 = 0.01$, $a_5 = 0.05$, and $g_4 = 0.01$ for three different cases: (i) $p = 1$, (ii) $p = 3$, and (iii) $p = 5$. The initial conditions for the cases (i) and (iii) are the same as those used in the left and right panels of Fig. 1, while for the case (ii) the initial conditions are chosen to be $\phi/M_{\text{pl}} = 9.0 \times 10^{-8}$, $v/M_{\text{pl}} = 5.0 \times 10^{-8}$, $\dot{v} = 0$, $m\phi/(\sqrt{6}M_{\text{pl}}H) = 1.0 \times 10^{-28}$, $r_\Sigma = 1.0 \times 10^{-12}$, and $1 - \Omega_r = 1.1 \times 10^{-8}$ at $z = 3.1 \times 10^{11}$. In these cases the conditions (4.8) are satisfied at $z = 10^{10}$, so w_{DE} starts to evolve from the value $1/3$ in the radiation era.

In all the cases shown in Fig. 3, the solutions finally approach the isotropic de Sitter fixed point with a vanishing anisotropic hair. We have also run numerical simulations by choosing other model parameters and found that the property of decreasing r_Σ and v after the radiation domination is generic. For $p = \mathcal{O}(1)$ the approach of w_{DE} to $w_{\text{DE}}^{(\text{iso})}$ occurs for $z \gg 1$, so the cosmological evolution at low redshifts is similar to that in the isotropic case.

B. $G_{2,3,4,5} \neq 0, G_6 \neq 0, g_4 = 0, g_5 \neq 0, f_{4,5,6} = 0$

If the terms g_5 and G_6 are present, they can modify the cosmological dynamics discussed in Sec. IV A. For concreteness, we consider the following functions

$$g_5(X) = \frac{2^{j_5-2}h_5}{mM_{\text{pl}}^{2j_5+1}}X^{j_5}, \quad G_6(X) = \frac{2^{p_6-1}h_6}{m^2M_{\text{pl}}^{2p_6}}X^{p_6}, \quad (4.19)$$

where h_5, j_5, h_6, p_6 are dimensionless constants. We assume that both j_5 and p_6 are positive.

In the early cosmological epoch, the main contribution to the dark energy density originating from the spatial component v corresponds to the term $-C_2H^2v^2$ in C_4 . The terms $H\phi g_5$ and H^2G_6 , which appear in C_2 as well as in q_V , can be expressed as

$$H\phi g_5 = h_5 \frac{\xi}{4} \left(\frac{\phi}{M_{\text{pl}}} \right)^{2j_5+1-p} \left(1 - \frac{v^2}{\phi^2} \right)^{j_5}, \quad (4.20)$$

$$H^2G_6 = h_6 \frac{\xi^2}{2} \left(\frac{\phi}{M_{\text{pl}}} \right)^{2p_6-2p} \left(1 - \frac{v^2}{\phi^2} \right)^{p_6}. \quad (4.21)$$

Provided that the parameter ξ given by Eq. (4.13) stays nearly constant around 1 and that $1 - v^2/\phi^2$ is at most of the order of 1, the conditions that the terms $H\phi g_5$ and H^2G_6 do not grow in the asymptotic past are given, respectively, by

$$j_5 \geq \frac{1}{2}(p-1), \quad p_6 \geq p. \quad (4.22)$$

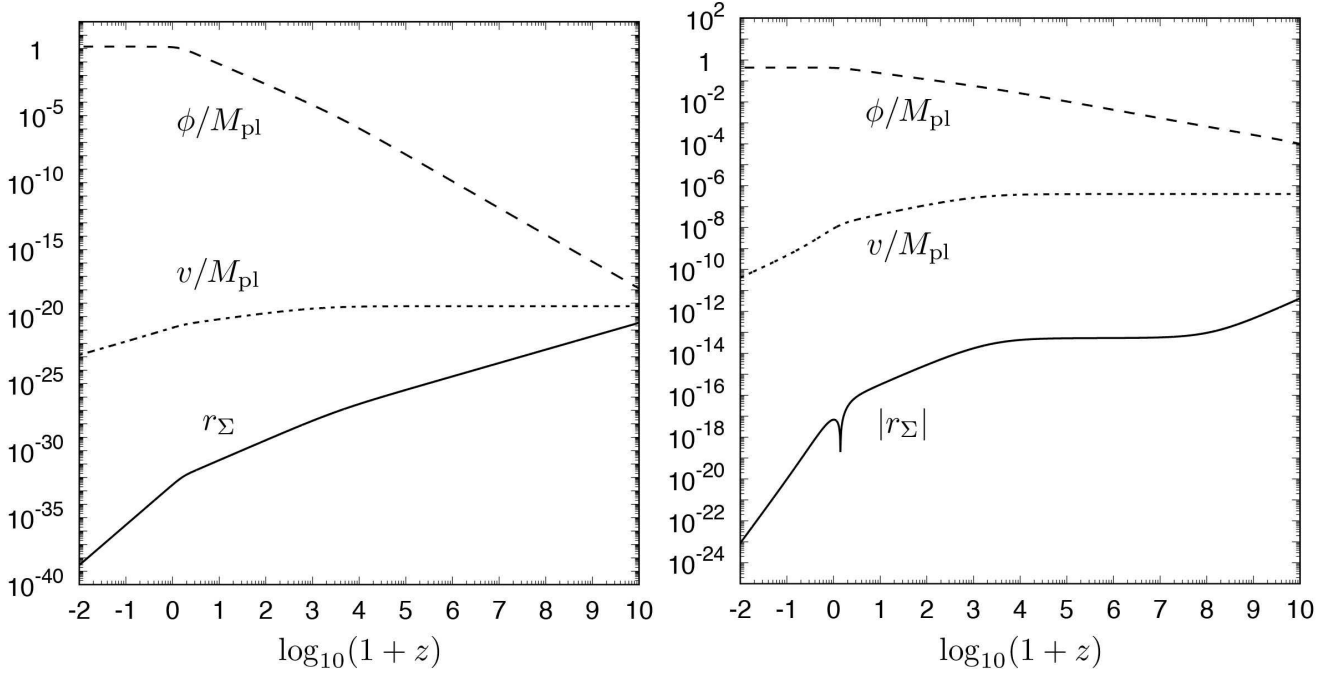


FIG. 4: Evolution of ϕ/M_{pl} , v/M_{pl} and r_Σ in the model (B) for the parameters $p = 1$, $a_4 = 0.01$, $a_5 = 0.05$, $j_5 = 1$, $h_5 = -0.4$, $G_6 = 0$ (left) and $p = 5$, $a_4 = 0.01$, $a_5 = 0.05$, $j_5 = 5$, $h_5 = -4.0 \times 10^{-5}$, $G_6 = 0$ (right). The initial conditions are chosen to be $\phi/M_{\text{pl}} = 6.0 \times 10^{-19}$, $v/M_{\text{pl}} = 6.0 \times 10^{-20}$, $\dot{v} = 0$, $m\phi/(\sqrt{6}M_{\text{pl}}H) = 2.0 \times 10^{-37}$, $r_\Sigma = 5.0 \times 10^{-21}$ and $1 - \Omega_r = 2.2 \times 10^{-7}$ at the redshift $z = 1.4 \times 10^{10}$ (left), and $\phi/M_{\text{pl}} = 6.0 \times 10^{-5}$, $v/M_{\text{pl}} = 4.0 \times 10^{-7}$, $\dot{v} = 0$, $m\phi/(\sqrt{6}M_{\text{pl}}H) = 2.0 \times 10^{-23}$, $r_\Sigma = 1.0 \times 10^{-11}$ and $1 - \Omega_r = 1.3 \times 10^{-7}$ at $z = 2.4 \times 10^{10}$ (right).

The energy densities corresponding to the terms $4H\phi g_5$ and $-3H^2G_6$ in C_2 are $\rho_{g_5} = -4\phi g_5 v^2 H^3$ and $\rho_{G_6} = 3G_6 v^2 H^4$, respectively, so the associated density parameters $\Omega_{g_5} = \rho_{g_5}/(3M_{\text{pl}}^2 H^2)$ and $\Omega_{G_6} = \rho_{G_6}/(3M_{\text{pl}}^2 H^2)$ read

$$\Omega_{g_5} = -\frac{4H\phi g_5}{3} \frac{v^2}{M_{\text{pl}}^2}, \quad \Omega_{G_6} = H^2 G_6 \frac{v^2}{M_{\text{pl}}^2}. \quad (4.23)$$

Under the conditions (4.22), we have that $|H\phi g_5| \ll 1$ and $|H^2 G_6| \ll 1$ in the early radiation era for the couplings h_5, h_6 at most of the order of unity. In this case, both $|\Omega_{g_5}|$ and $|\Omega_{G_6}|$ are much smaller than 1 for $v^2 \lesssim M_{\text{pl}}^2$. If the conditions (4.22) are violated, the terms $(\phi/M_{\text{pl}})^{2j_5+1-p}$ and $(\phi/M_{\text{pl}})^{2p_6-2p}$ grow as we go back to the past. This leads to the values of $|\Omega_{g_5}|$ and $|\Omega_{G_6}|$ larger than 1 (unless we choose very small values of $|h_5|$ and $|h_6|$). To avoid this behavior, we need to impose the conditions (4.22).

Provided that $|H\phi g_5| \ll 1$ and $|H^2 G_6| \ll 1$, the quantity q_V is close to 1 with $|\alpha_1/q_V| \ll 1$ in Eq. (3.14). Moreover the g_5 and G_6 dependent terms in α_2 are suppressed relative to q_V , so the expression of α_2 is similar to that given in Eq. (4.17). As long as $|\alpha_2/q_V| \ll 1$, the evolution of v during the radiation era should be given by $v = \text{constant}$. As for the anisotropic expansion rate during the radiation era, we have that $C_2 \simeq -1/2$ and $|H^2 C_5| \ll |H C_2|$ in Eq. (3.8) under the conditions $|H\phi g_5| \ll 1$ and $|H^2 G_6| \ll 1$, so we obtain the solution in the form (4.18) with $q_V \simeq 1$. This means that, unless $q_V v^2/(3M_{\text{pl}}^2)$ is much smaller than $|\mathcal{B}/(a^3 H)|$ in the radiation-dominated epoch, r_Σ temporally approaches the constant value $q_V v^2/(3M_{\text{pl}}^2)$. If the condition $q_V v^2/(3M_{\text{pl}}^2) \ll |\mathcal{B}/(a^3 H)|$ is always satisfied in the radiation era, then r_Σ decreases as $r_\Sigma \propto a^{-1}$.

In Fig. 4 we plot two examples for the evolution of ϕ, v, r_Σ with the model parameters $h_5 = -0.4$, $j_5 = 1$, $G_6 = 0$, and $p = 1$ (left) and $h_5 = -4.0 \times 10^{-5}$, $j_5 = 5$, $G_6 = 0$, and $p = 5$ (right). The temporal vector component ϕ always increases during the radiation and matter eras. As estimated analytically, the spatial vector component v stays nearly constant in the radiation era and it decreases in proportion to $a^{-1/2}$ during the matter dominance. In the left panel of Fig. 4, the initial value of v is small such that the condition $q_V v^2/(3M_{\text{pl}}^2) \ll |\mathcal{B}/(a^3 H)|$ is satisfied, so the ratio r_Σ decreases as $\propto a^{-1}$ in the radiation era. In the right panel of Fig. 4, the initial value of v is larger than that for $p = 1$, so r_Σ temporally approaches the constant value $q_V v^2/(3M_{\text{pl}}^2)$. In both cases, r_Σ decreases after the end of the radiation era. In the numerical simulations of Fig. 4 the conditions (4.8) are initially satisfied with $\mathcal{C}_{10} \simeq 2\mathcal{C}_2$, so the

vector field temporally behaves as a dark radiation ($w_{\text{DE}} \simeq 1/3$) until w_{DE} approaches the isotropic value $w_{\text{DE}}^{(\text{iso})}$. As in the case of the model (A), the approach of w_{DE} to $w_{\text{DE}}^{(\text{iso})}$ occurs at high redshifts ($z \gg 1$) for $p = \mathcal{O}(1)$.

When $g_5 < 0$, the quantity \mathcal{A}_V defined by Eq. (3.36) is positive, so there exists the anisotropic de Sitter fixed point characterized by Eq. (3.41). As we showed analytically in Sec III B, this anisotropic fixed point is not stable, while the isotropic de Sitter one is stable. In fact, the numerical simulation of Fig. 4 (which corresponds to $g_5 < 0$) shows that the solution finally approaches the isotropic de Sitter fixed point characterized by $v = 0$ and $\Sigma = 0$, so the anisotropic hair does not survive.

We also carry out numerical simulations for the model with non-zero G_6 and find that the cosmological evolution is qualitatively similar to that for $g_5 \neq 0$. The general result is that, for cosmologically viable models with the late-time acceleration, the spatial vector component evolves as $v = \text{constant}$ (radiation era), $v \propto a^{-1/2}$ (matter era), $v \propto a^{-1}$ (de Sitter era) and that the ratio r_Σ finally approaches 0. The evolution of w_{DE} at low redshifts is similar to that in the isotropic case.

C. $f_{4,5,6} \neq 0$

Let us finally study the beyond-generalized Proca theories described by the action (2.1) with $\mathcal{L}^N \neq 0$. Since $f_4 \neq 0$ and $f_5 \neq 0$ in such theories, the existence of coefficient C_1 in Eqs. (2.36)-(2.39) leads to the dynamical system (4.15) with different Z , \mathbf{x} , and \mathbf{y} . In this case, after taking the time derivative of Eq. (2.36), we first eliminate the second derivative $\ddot{\phi}$ on account of Eq. (2.37). Similarly, the $\ddot{\phi}$ term is eliminated by combining Eq. (2.37) with Eq. (2.38). On using these two equations with Eq. (2.40), we can derive the two equations for $\dot{\Sigma}$ without containing \ddot{v} . Eliminating the $\dot{\Sigma}$ term, we obtain the following equation

$$v \mathcal{F}(\phi, \dot{\phi}, v, \dot{v}, H, \Sigma) = 0, \quad (4.24)$$

where \mathcal{F} is a function that depends on the quantities inside the parenthesis. One of the branches of solutions is given by

$$v = 0. \quad (4.25)$$

Another branch corresponds to $\mathcal{F} = 0$, which gives rise to a non-vanishing value of v . Recall that the coefficient C_1 given in Eq. (2.30) contains the term v^2 . If $v \neq 0$, then we can express $\dot{\phi}$ in terms of $\phi, v, \dot{v}, H, \Sigma$ by using Eq. (2.36). Taking the time derivative of Eq. (2.36) and using Eqs. (2.37) and (2.38), we obtain two equations after the elimination of $\ddot{\phi}$. On using other equations as well, the dynamical equations of motion can be written in the form (4.15), where $\mathbf{x} = {}^t(\ddot{v}, \dot{H}, \dot{\Sigma})$, and Z, \mathbf{y} are the 3×3 and 1×3 matrices, respectively, involving the dependence of $\dot{v}, v, \phi, H, \Sigma$. Unlike the theories with $f_{4,5} = 0$, the $\dot{\phi}$ term is determined by the constraint equation (2.36). Computing the determinant of the matrix Z , we find that the determinant vanishes exactly. When the determinant vanishes we cannot solve Eq. (4.15) in the form $\mathbf{x} = Z^{-1}\mathbf{y}$, so the dynamical system does not reduce to the closed autonomous system. Note that this kind of determinant singularity also appears in the context of anisotropic string cosmology with dilaton and axion fields [39].

The above discussion shows that only the branch $v = 0$ is physically allowed for the theories with $f_{4,5} \neq 0$. For this branch all the terms containing C_1 in Eqs. (2.36)-(2.39) vanish, so the dynamical system is similar to that of second-order generalized Proca theories with $v = 0$. Then the anisotropic expansion rate simply decreases as $\Sigma \propto a^{-3}$. If the ratio $|r_\Sigma|$ is much smaller than 1 at the onset of the radiation domination, the effect of Σ on the dynamical equations of motion is negligible during most of the cosmic expansion history. In this sense, the cosmological dynamics for the theories with $f_{4,5} \neq 0$ is very similar to that of the isotropic case without having the C_1 -dependent terms.

V. CONCLUSIONS

In beyond-generalized Proca theories, we have studied the anisotropic cosmological dynamics in the presence of a spatial vector component v . On the isotropic FLRW background it was found in Ref. [32] that, even with the Lagrangian density \mathcal{L}^N outside the domain of second-order generalized Proca theories, there is no additional DOF associated with the Ostrogradski ghost. In this paper we showed that the same result also holds on the anisotropic background. There exists the constraint equation (2.27) related with the Hamiltonian \mathcal{H} as Eq. (2.34), so that $\mathcal{H} = 0$. Hence the beyond-generalized Proca theories are free from the Ostrogradski instability with the Hamiltonian unbounded from below.

In Sec. III A we analytically estimated the evolution of the anisotropic expansion rate Σ and the spatial component v in the early cosmological epoch. If the conditions (3.9) hold in the radiation and matter eras, Σ decreases in proportion to a^{-3} . Under the conditions (3.18), the evolution of v is given by $v = \text{constant}$ during the radiation era and $v \propto a^{-1/2}$ during the matter era. If v is not very much smaller than M_{pl} during the radiation domination, there are cases in which the second condition of Eq. (3.9) is violated. In concrete dark energy models studied in Sec. IV, we showed the existence of solutions on which the ratio $r_\Sigma = \Sigma/H$ remains nearly constant during the radiation era.

In Sec. III B we discussed the property of de Sitter fixed points relevant to the late-time cosmic acceleration. Besides the isotropic point (3.29), we found the existence of anisotropic fixed points (3.41) under the two conditions (3.32) and (3.33). For the theories in which the parameter \mathcal{A}_V defined by Eq. (3.36) vanishes, we only have the isotropic fixed point. For $\mathcal{A}_V \neq 0$ the anisotropic fixed points exist, but they are not stable. In both cases, the analytic estimation implies that the solutions approach the stable isotropic point in accordance with the cosmic no-hair conjecture.

In Sec. IV we studied the evolution of anisotropic cosmological solutions in a class of dark energy models given by the functions (4.1). In the early cosmological epoch, the contributions of v and Σ to the energy density ρ_{DE} and the pressure P_{DE} can be larger than the isotropic contributions associated with the temporal vector component ϕ . If v is large such that the conditions (4.8) are satisfied, the dark energy equation of state is given by Eq. (4.9) in the radiation era, which is close to $w_{\text{DE}} = 1/3$ in concrete models studied in Secs. IV A and IV B. If the contribution of Σ dominates over that of v such that the conditions (4.10) are satisfied, we have $w_{\text{DE}} \simeq 1$ during the radiation era. In both cases, the dark energy equation of state is different from the isotropic value $w_{\text{DE}}^{(\text{iso})}$ given by Eq. (4.3). However, the transition of w_{DE} to the value $w_{\text{DE}}^{(\text{iso})}$ typically occurs at high redshifts (see Figs. 2 and 3), so the dark energy dynamics at low redshifts is similar to that in the isotropic case.

In generalized Proca theories with v not very much smaller than M_{pl} , the spatial anisotropy in the radiation era can be sustained by v with the nearly constant ratio $r_\Sigma \simeq q_V v^2 / (3M_{\text{pl}}^2)$. In this regime, for the models (A) and (B) studied in Sec. IV, the vector field behaves as a dark radiation characterized by $w_{\text{DE}} \simeq 1/3$. As seen in the right panels of Fig. 1 and 4, the constant behavior of r_Σ in the radiation era can occur for the models with large powers p (like $p = 5$) due to the possible choice of large initial values of v . On the other hand, for the models with small p (like $p = 1$), we have $q_V v^2 / (3M_{\text{pl}}^2) \ll |\mathcal{B}/(a^3 H)|$ in Eq. (4.18) and hence r_Σ decreases as $\propto a^{-1}$ during the radiation era (see the left panels of Fig. 1 and 4). After the matter dominance, both v and Σ decrease toward the isotropic fixed point ($v = 0 = \Sigma$).

In beyond-generalized Proca theories, we showed that the physical branch of solutions without having a determinant singularity of the dynamical system corresponds to $v = 0$. In this case the anisotropic expansion rate simply decreases as $\Sigma \propto a^{-3}$ from the onset of the radiation-dominated epoch, so the cosmological evolution is practically indistinguishable from the isotropic case. Interestingly, the beyond-generalized Proca theories do not allow the existence of anisotropic solutions with constant r_Σ .

We have thus shown that, apart from the radiation era in the presence of a non-negligible spatial vector component v , the anisotropy does not survive for a class of dark energy models in the framework of (beyond-)generalized Proca theories. Thus, the analysis of Refs. [30, 31] where the spatial component was treated as a perturbation on the isotropic FLRW background can be justified except for the early cosmological epoch in which the vector field behaves as a dark radiation. It will be of interest to place detailed observational constraints on both isotropic and anisotropic dark energy models from the observations of CMB, type Ia supernovae, and large-scale structures.

Acknowledgements

LH thanks financial support from Dr. Max Rössler, the Walter Haefner Foundation and the ETH Zurich Foundation. RK is supported by the Grant-in-Aid for Research Activity Start-up of the JSPS No. 15H06635. ST is supported by the Grant-in-Aid for Scientific Research Fund of the JSPS No. 16K05359 and MEXT KAKENHI Grant-in-Aid for Scientific Research on Innovative Areas ‘‘Cosmic Acceleration’’ (No. 15H05890).

Appendix A: Coefficients of equations of motion

The coefficients appearing in Eqs. (2.37)-(2.40) are given by

$$\begin{aligned}
C_5 &= 4\phi g_5 - 4(H + \Sigma) [G_6 + \phi^2(G_{6,X} + 2f_6)] , \\
C_6 &= 2v [C_2 - 6\phi\Sigma g_5 - 2G_{4,X} - 2\phi^2 f_4 - (H + \Sigma) \{ \phi(6\phi^2 f_5 - G_{5,X}) - 6\Sigma(G_6 + \phi^2 G_{6,X} + 2\phi^2 f_6) \}] , \\
C_7 &= 12\phi^2 v^2 f_5 , \quad C_8 = -2 [G_6 + \phi^2(G_{6,X} + 2f_6)] , \\
C_9 &= 2v [\phi(4g_5 + G_{5,X} - 6\phi^2 f_5) - 6H(G_6 + \phi^2 G_{6,X} + 2\phi^2 f_6)] , \\
C_{10} &= 12(G_4 - \phi^2 G_{4,X} - \phi^4 f_4) + 6\phi^3 H(G_{5,X} - 6\phi^2 f_5) \\
&\quad - v^2 [1 - 2g_4 - 12\phi^2 f_4 - 36H\phi^3 f_5 - 12\phi(H - \Sigma)g_5 + 6(2H^2 - 2H\Sigma - \Sigma^2)(G_6 + \phi^2 G_{6,X} + 2\phi^2 f_6)] , \\
C_{11} &= 2v [\phi(G_{5,X} - 2g_5 - 6\phi^2 f_5) + 6\Sigma(G_6 + \phi^2 G_{6,X} + 2\phi^2 f_6)] , \\
C_{12} &= 6\phi^3 \Sigma(6\phi^2 f_5 - G_{5,X}) + 2v^2 [1 - 2g_4 - 18\phi^3 \Sigma f_5 - 6\phi H g_5 + 3(H^2 + 2H\Sigma - 2\Sigma^2)(G_6 + \phi^2 G_{6,X} + 2\phi^2 f_6)] , \\
C_{13} &= 4v^2 [f_4 + \phi^2 f_{4,X} + 3\phi(H + \Sigma)(2f_5 + \phi^2 f_{5,X})] , \\
C_{14} &= 2g_5 + 2\phi^2 g_{5,X} - 2\phi(H + \Sigma) [3G_{6,X} + 4f_6 + \phi^2(G_{6,XX} + 2f_{6,X})] , \\
C_{15} &= 2v [\phi g_{4,X} + 2(2H - \Sigma)(g_5 + \phi^2 g_{5,X}) - 2\phi(G_{4,XX} + \phi^2 f_{4,X}) \\
&\quad + (H + \Sigma) \{ G_{5,X} + \phi^2 G_{5,XX} - 6\phi^2(f_5 + \phi^2 f_{5,X}) \} \\
&\quad - 3\phi(H^2 - \Sigma^2)(3G_{6,X} + \phi^2 G_{6,XX} + 4f_6 + 2\phi^2 f_{6,X}) - 2\phi v^2 \{ f_{4,X} + 3\phi(H + \Sigma)f_{5,X} \}] , \\
C_{16} &= -3\phi^2 G_{3,X} - 12\phi H [G_{4,X} + \phi^2 G_{4,XX} + \phi^2(4f_4 + \phi^2 f_{4,X})] \\
&\quad + 3\phi^2(H^2 - \Sigma^2) [3G_{5,X} + \phi^2 G_{5,XX} - 6\phi^2(5f_5 + \phi^2 f_{5,X})] \\
&\quad + v^2 [2(H - 2\Sigma) \{ \phi g_{4,X} + 3H(g_5 + \phi^2 g_{5,X}) \} + 12\phi \{ (2H - \Sigma)f_4 + H\phi^2 f_{4,X} \} \\
&\quad + 18\phi^2(H^2 - \Sigma^2)(4f_5 + \phi^2 f_{5,X}) - 2\phi(2H - \Sigma)(H - 2\Sigma)(H + \Sigma)(3G_{6,X} + \phi^2 G_{6,XX} + 4f_6 + 2\phi^2 f_{6,X})] , \\
C_{17} &= -2v [\phi g_{5,X} - (H + \Sigma) \{ G_{6,X} + \phi^2(G_{6,XX} + 2f_{6,X}) \}] , \\
C_{18} &= 2v^2 [2G_{4,XX} - g_{4,X} + 2\phi^2 f_{4,X} + \phi(H + \Sigma)(6\phi^2 f_{5,X} - G_{5,XX}) - (H + \Sigma)^2(G_{6,X} + \phi^2 G_{6,XX} + 2\phi^2 f_{6,X})] \\
&\quad + 2v(2H - \Sigma)C_{17} - C_2 + \frac{3}{2}(2H - \Sigma)C_5 - 4G_{4,X} - 4\phi^2 f_4 - 2\phi(H + \Sigma)(6\phi^2 f_5 - G_{5,X}) , \\
C_{19} &= -2v^3 [6H\phi^2 f_{4,X} + 9\phi^3(H^2 - \Sigma^2)f_{5,X} + (H - 2\Sigma) \{ g_{4,X} + (H + \Sigma)^2(G_{6,X} + \phi^2 G_{6,XX} + 2\phi^2 f_{6,X}) \}] \\
&\quad + 3v^2 H(H - 2\Sigma)C_{17} + v [3\phi G_{3,X} + 4(H + \Sigma)C_2 + 2(H^2 - 7H\Sigma + \Sigma^2)C_5 - 12(H - \Sigma)G_{4,X} \\
&\quad + 12\phi^2(2H + \Sigma)f_4 + 12\phi^2 H(\phi^2 f_{4,X} + G_{4,XX}) + 3\phi(H^2 - \Sigma^2) \{ G_{5,X} - \phi^2(G_{5,XX} - 6f_5 - 6\phi^2 f_{5,X}) \}] , \\
C_{20} &= 3G_2 + 18(H^2 + \Sigma^2)(G_4 - \phi^2 G_{4,X} - \phi^4 f_4) - 6\phi^3(H^3 + \Sigma^3)(6\phi^2 f_5 - G_{5,X}) \\
&\quad + v^2 \left[3(H^2 - 4\Sigma^2)C_2 - \frac{3}{2}\Sigma(5H + 2\Sigma)(H - 2\Sigma)C_5 + 18\phi^2(H^2 + \Sigma^2)f_4 + 36\phi^3(H^3 + \Sigma^3)f_5 \right] , \\
C_{21} &= [3(H - 2\Sigma)C_5 - 12G_{4,X} - 12\phi^2 f_4 - 6\phi(H + \Sigma)(6\phi^2 f_5 - G_{5,X})] v - 2C_6 , \\
C_{22} &= v^2 \left[4(2\Sigma - H)C_2 + \frac{1}{2}(5H + 2\Sigma)(H - 2\Sigma)C_5 - 12\phi^2 \Sigma \{ f_4 + 3\phi(H + \Sigma)f_5 \} \right] \\
&\quad - 6\Sigma [2G_4 - 2\phi^2 G_{4,X} - 2\phi^4 f_4 + \phi^3(H + \Sigma)(G_{5,X} - 6\phi^2 f_5)] , \\
D_1 &= -\phi g_{4,X} - 2(H + \Sigma)(g_5 + \phi^2 g_{5,X}) + \phi(H + \Sigma)^2(3G_{6,X} + \phi^2 G_{6,XX} + 4f_6 + 2\phi^2 f_{6,X}) , \\
D_2 &= v [G_{3,X} - 2\phi(H - 2\Sigma)g_{4,X} + (H + \Sigma) \{ 4\phi(G_{4,XX} + 4f_4 + \phi^2 f_{4,X}) - 4(H - 2\Sigma)(g_5 + \phi^2 g_{5,X}) \} \\
&\quad - (H + \Sigma)^2 \{ G_{5,X} + \phi^2 G_{5,XX} - 30\phi^2 f_5 - 6\phi^4 f_{5,X} - 2\phi(H - 2\Sigma)(3G_{6,X} + \phi^2 G_{6,XX} + 4f_6 + 2\phi^2 f_{6,X}) \}] \\
&\quad - 2\phi v^3 (H + \Sigma) [2f_{4,X} + 3\phi(H + \Sigma)f_{5,X}] , \\
D_3 &= \phi(G_{2,X} + 3H\phi G_{3,X}) + 6\phi(H^2 - \Sigma^2)(G_{4,X} + \phi^2 G_{4,XX} + 4\phi^2 f_4 + \phi^4 f_{4,X}) \\
&\quad - \phi^2(H + \Sigma)^2(H - 2\Sigma)(3G_{5,X} + \phi^2 G_{5,XX} - 30\phi^2 f_5 - 6\phi^4 f_{5,X}) \\
&\quad - v^2 [(H - 2\Sigma)^2 \{ \phi g_{4,X} + 2(H + \Sigma)(g_5 + \phi^2 g_{5,X}) \} - 6\phi(H + \Sigma) \{ 2\Sigma f_4 - (H - \Sigma)\phi^2 f_{4,X} \} \\
&\quad - (H + \Sigma)^2 \{ 6\phi^2(6\Sigma f_5 - \phi^2(H - 2\Sigma)f_{5,X}) + \phi(H - 2\Sigma)^2(3G_{6,X} + \phi^2 G_{6,XX} + 4f_6 + 2\phi^2 f_{6,X}) \}] , \\
D_4 &= 1 - 2g_4 - 4(H + \Sigma)\phi g_5 + 2(H + \Sigma)^2(G_6 + \phi^2 G_{6,X} + 2\phi^2 f_6) , \\
D_5 &= v [g_{4,X} + 2\phi(H + \Sigma)g_{5,X} - (H + \Sigma)^2(G_{6,X} + \phi^2 G_{6,XX} + 2\phi^2 f_{6,X})] ,
\end{aligned}$$

$$\begin{aligned}
D_6 = & v[G_{2,X} + 3\phi HG_{3,X} + 2(H + \Sigma)(H - 2\Sigma)\{1 - 2g_4 - 4\phi(H + \Sigma)g_5\} \\
& + 6(H + \Sigma)\{(H + \Sigma)G_{4,X} + 2\phi^2(2H - \Sigma)f_4 + \phi^2(H - \Sigma)(G_{4,XX} + \phi^2 f_{4,X})\} \\
& - \phi(H + \Sigma)^2\{3HG_{5,X} - 6\phi^2(5H - 4\Sigma)f_5 + \phi^2(H - 2\Sigma)(G_{5,XX} - 6\phi^2 f_{5,X})\} \\
& + 4(H + \Sigma)^3(H - 2\Sigma)(G_6 + \phi^2 G_{6,X} + 2\phi^2 f_6) - v^3[(H - 2\Sigma)^2(g_{4,X} + 2\phi(H + \Sigma)g_{5,X}) \\
& + 6\phi^2(H^2 - \Sigma^2)f_{4,X} + (H + \Sigma)^2(H - 2\Sigma)\{6\phi^3 f_{5,X} - (H - 2\Sigma)(G_{6,X} + \phi^2 G_{6,XX} + 2\phi^2 f_{6,X})\}]. \quad (A1)
\end{aligned}$$

-
- [1] D. N. Spergel *et al.* [WMAP Collaboration], *Astrophys. J. Suppl.* **148**, 175 (2003) [astro-ph/0302209]; P. A. R. Ade *et al.* [Planck Collaboration], *Astron. Astrophys.* **571**, A16 (2014) [arXiv:1303.5076 [astro-ph.CO]].
- [2] M. Tegmark *et al.* [SDSS Collaboration], *Phys. Rev. D* **69**, 103501 (2004) [astro-ph/0310723]; M. Tegmark *et al.* [SDSS Collaboration], *Phys. Rev. D* **74**, 123507 (2006) [astro-ph/0608632].
- [3] A. G. Riess *et al.* [Supernova Search Team Collaboration], *Astron. J.* **116**, 1009 (1998) [astro-ph/9805201]; S. Perlmutter *et al.* [Supernova Cosmology Project Collaboration], *Astrophys. J.* **517**, 565 (1999) [astro-ph/9812133].
- [4] S. Weinberg, *Rev. Mod. Phys.* **61**, 1 (1989).
- [5] C. de Rham and G. Gabadadze, *Phys. Rev. D* **82**, 044020 (2010) [arXiv:1007.0443 [hep-th]]; C. de Rham, G. Gabadadze and A. J. Tolley, *Phys. Rev. Lett.* **106**, 231101 (2011) [arXiv:1011.1232 [hep-th]]; C. de Rham, G. Gabadadze, L. Heisenberg and D. Pirtskhalava, *Phys. Rev. D* **83**, 103516 (2011) [arXiv:1010.1780 [hep-th]].
- [6] C. Wetterich, *Gen. Rel. Grav.* **30**, 159 (1998) [gr-qc/9704052]; A. O. Barvinsky, *Phys. Lett. B* **572**, 109 (2003) [hep-th/0304229]; S. Deser and R. P. Woodard, *Phys. Rev. Lett.* **99**, 111301 (2007) [arXiv:0706.2151 [astro-ph]]; M. Jaccard, M. Maggiore and E. Mitsou, *Phys. Rev. D* **88**, 044033 (2013) [arXiv:1305.3034 [hep-th]]; L. Modesto and S. Tsujikawa, *Phys. Lett. B* **727**, 48 (2013) [arXiv:1307.6968 [hep-th]].
- [7] E. J. Copeland, M. Sami and S. Tsujikawa, *Int. J. Mod. Phys. D* **15**, 1753 (2006) [hep-th/0603057]; A. Silvestri and M. Trodden, *Rept. Prog. Phys.* **72**, 096901 (2009) [arXiv:0904.0024 [astro-ph.CO]]; S. Tsujikawa, arXiv:1004.1493 [astro-ph.CO]; T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, *Phys. Rept.* **513**, 1 (2012) [arXiv:1106.2476 [astro-ph.CO]]; S. Tsujikawa, *Lect. Notes Phys.* **800**, 99 (2010) [arXiv:1101.0191 [gr-qc]]; A. Joyce, B. Jain, J. Khoury and M. Trodden, *Phys. Rept.* **568**, 1 (2015) [arXiv:1407.0059 [astro-ph.CO]]; P. Bull *et al.*, *Phys. Dark Univ.* **12**, 56 (2016) [arXiv:1512.05356 [astro-ph.CO]].
- [8] Y. Fujii, *Phys. Rev. D* **26**, 2580 (1982); L. H. Ford, *Phys. Rev. D* **35**, 2339 (1987); C. Wetterich, *Nucl. Phys. B* **302**, 668 (1988); B. Ratra and P. J. E. Peebles, *Phys. Rev. D* **37**, 3406 (1988); T. Chiba, N. Sugiyama and T. Nakamura, *Mon. Not. Roy. Astron. Soc.* **289**, L5 (1997) [astro-ph/9704199]; P. G. Ferreira and M. Joyce, *Phys. Rev. Lett.* **79**, 4740 (1997) [astro-ph/9707286]; R. R. Caldwell, R. Dave and P. J. Steinhardt, *Phys. Rev. Lett.* **80**, 1582 (1998) [astro-ph/9708069].
- [9] C. Brans and R. H. Dicke, *Phys. Rev.* **124**, 925 (1961); Y. Fujii and K. Maeda, “The Scalar-Tensor Theory of Gravitation”, Cambridge University Press (2003).
- [10] A. Nicolis, R. Rattazzi and E. Trincherini, *Phys. Rev. D* **79**, 064036 (2009) [arXiv:0811.2197 [hep-th]].
- [11] C. Deffayet, G. Esposito-Farese and A. Vikman, *Phys. Rev. D* **79**, 084003 (2009) [arXiv:0901.1314 [hep-th]]; C. Deffayet, S. Deser and G. Esposito-Farese, *Phys. Rev. D* **80**, 064015 (2009) [arXiv:0906.1967 [gr-qc]]; C. de Rham and L. Heisenberg, *Phys. Rev. D* **84**, 043503 (2011) [arXiv:1106.3312 [hep-th]]; L. Heisenberg, R. Kimura and K. Yamamoto, *Phys. Rev. D* **89**, 103008 (2014) doi:10.1103/PhysRevD.89.103008 [arXiv:1403.2049 [hep-th]].
- [12] G. W. Horndeski, *Int. J. Theor. Phys.* **10**, 363-384 (1974).
- [13] C. Deffayet, X. Gao, D. A. Steer and G. Zahariade, *Phys. Rev. D* **84**, 064039 (2011) [arXiv:1103.3260 [hep-th]]; T. Kobayashi, M. Yamaguchi and J. Yokoyama, *Prog. Theor. Phys.* **126**, 511 (2011) [arXiv:1105.5723 [hep-th]]; C. Charmousis, E. J. Copeland, A. Padilla and P. M. Saffin, *Phys. Rev. Lett.* **108**, 051101 (2012) [arXiv:1106.2000 [hep-th]].
- [14] J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi, *Phys. Rev. Lett.* **114**, 211101 (2015) [arXiv:1404.6495 [hep-th]].
- [15] C. Lin, S. Mukohyama, R. Namba and R. Saitou, *JCAP* **1410**, 071 (2014) [arXiv:1408.0670 [hep-th]]; X. Gao, *Phys. Rev. D* **90**, 104033 (2014) [arXiv:1409.6708 [gr-qc]]; C. Deffayet, G. Esposito-Farese and D. A. Steer, *Phys. Rev. D* **92**, 084013 (2015) [arXiv:1506.01974 [gr-qc]].
- [16] L. A. Gergely and S. Tsujikawa, *Phys. Rev. D* **89**, 064059 (2014) [arXiv:1402.0553 [hep-th]]; R. Kase and S. Tsujikawa, *Phys. Rev. D* **90**, 044073 (2014) [arXiv:1407.0794 [hep-th]]; J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi, *JCAP* **1502**, 018 (2015) [arXiv:1408.1952 [astro-ph.CO]]; R. Kase and S. Tsujikawa, *Int. J. Mod. Phys. D* **23**, no. 13, 1443008 (2015) [arXiv:1409.1984 [hep-th]]; A. De Felice, K. Koyama and S. Tsujikawa, *JCAP* **1505**, no. 05, 058 (2015) [arXiv:1503.06539 [gr-qc]]; J. Gleyzes, D. Langlois, M. Mancarella and F. Vernizzi, *JCAP* **1508**, 054 (2015) [arXiv:1504.05481 [astro-ph.CO]]; T. Kobayashi, M. Yamaguchi and J. Yokoyama, *JCAP* **1507**, 017 (2015) [arXiv:1504.05710 [hep-th]]; R. Kase, S. Tsujikawa and A. De Felice, *Phys. Rev. D* **93**, 024007 (2016) [arXiv:1510.06853 [gr-qc]].
- [17] R. Kase, L. A. Gergely and S. Tsujikawa, *Phys. Rev. D* **90**, 124019 (2014) [arXiv:1406.2402 [hep-th]]; T. Kobayashi, Y. Watanabe and D. Yamauchi, *Phys. Rev. D* **91**, 064013 (2015) [arXiv:1411.4130 [gr-qc]]; K. Koyama and J. Sakstein, *Phys. Rev. D* **91**, 124066 (2015) [arXiv:1502.06872 [astro-ph.CO]]; A. De Felice, R. Kase and S. Tsujikawa, *Phys. Rev. D* **92**, 124060 (2015) [arXiv:1508.06364 [gr-qc]]; R. Kase, S. Tsujikawa and A. De Felice, *JCAP* **1603**, 003 (2016) [arXiv:1512.06497 [gr-qc]]; E. Babichev, K. Koyama, D. Langlois, R. Saito and J. Sakstein, arXiv:1606.06627 [gr-qc].
- [18] J. D. Barrow, M. Thorsrud and K. Yamamoto, *JHEP* **1302**, 146 (2013) [arXiv:1211.5403 [gr-qc]].

- [19] J. Beltran Jimenez and A. L. Maroto, Phys. Rev. D **78**, 063005 (2008) doi:10.1103/PhysRevD.78.063005 [arXiv:0801.1486 [astro-ph]]; J. Beltran Jimenez and A. L. Maroto, JCAP **0903**, 016 (2009) doi:10.1088/1475-7516/2009/03/016 [arXiv:0811.0566 [astro-ph]]; J. Beltran Jimenez and A. L. Maroto, Phys. Rev. D **80**, 063512 (2009) doi:10.1103/PhysRevD.80.063512 [arXiv:0905.1245 [astro-ph.CO]]; J. B. Jimenez, R. Durrer, L. Heisenberg and M. Thorsrud, JCAP **1310**, 064 (2013) [arXiv:1308.1867 [hep-th]]; J. Beltran Jimenez and T. S. Koivisto, Phys. Lett. B **756**, 400 (2016) doi:10.1016/j.physletb.2016.03.047 [arXiv:1509.02476 [gr-qc]]; J. Beltran Jimenez, L. Heisenberg and T. S. Koivisto, JCAP **1604**, no. 04, 046 (2016) doi:10.1088/1475-7516/2016/04/046 [arXiv:1602.07287 [hep-th]].
- [20] G. Tasinato, K. Koyama and N. Khosravi, JCAP **1311**, 037 (2013) [arXiv:1307.0077 [hep-th]].
- [21] P. Fleury, J. P. B. Almeida, C. Pitrou and J. P. Uzan, JCAP **1411**, 043 (2014). [arXiv:1406.6254 [hep-th]].
- [22] M. Hull, K. Koyama and G. Tasinato, JHEP **1503**, 154 (2015) [arXiv:1408.6871 [hep-th]]; M. Hull, K. Koyama and G. Tasinato, Phys. Rev. D **93**, 064012 (2016) [arXiv:1510.07029 [hep-th]].
- [23] C. Deffayet, A. E. Gumrukcuoglu, S. Mukohyama and Y. Wang, JHEP **1404**, 082 (2014). [arXiv:1312.6690 [hep-th]]; C. Deffayet, S. Mukohyama and V. Sivanesan, Phys. Rev. D **93**, 085027 (2016) [arXiv:1601.01287 [hep-th]].
- [24] L. Heisenberg, JCAP **1405**, 015 (2014) [arXiv:1402.7026 [hep-th]].
- [25] G. Tasinato, JHEP **1404**, 067 (2014) [arXiv:1402.6450 [hep-th]]; G. Tasinato, Class. Quant. Grav. **31**, 225004 (2014) [arXiv:1404.4883 [hep-th]].
- [26] E. Allys, P. Peter and Y. Rodriguez, JCAP **1602**, 004 (2016) [arXiv:1511.03101 [hep-th]]; E. Allys, J. P. B. Almeida, P. Peter and Y. Rodriguez, arXiv:1605.08355 [hep-th].
- [27] J. B. Jimenez and L. Heisenberg, Phys. Lett. B **757**, 405 (2016) [arXiv:1602.03410 [hep-th]].
- [28] A. De Felice, L. Heisenberg, R. Kase, S. Tsujikawa, Y. I. Zhang and G. B. Zhao, Phys. Rev. D **93**, 104016 (2016) [arXiv:1602.00371 [gr-qc]].
- [29] J. Chagoya, G. Niz and G. Tasinato, arXiv:1602.08697 [hep-th].
- [30] A. De Felice, L. Heisenberg, R. Kase, S. Mukohyama, S. Tsujikawa and Y. I. Zhang, JCAP **1606**, 048 (2016) [arXiv:1603.05806 [gr-qc]].
- [31] A. De Felice, L. Heisenberg, R. Kase, S. Mukohyama, S. Tsujikawa and Y. I. Zhang, arXiv:1605.05066 [gr-qc].
- [32] L. Heisenberg, R. Kase and S. Tsujikawa, arXiv:1605.05565 [hep-th].
- [33] M. V. Ostrogradski, Mem. Acad. St. Petersburg VI 4, **385** (1850); R. P. Woodard, Lect. Notes Phys. **720**, 403 (2007) [astro-ph/0601672].
- [34] M. a. Watanabe, S. Kanno and J. Soda, Phys. Rev. Lett. **102**, 191302 (2009) [arXiv:0902.2833 [hep-th]]; N. Bartolo, S. Matarrese, M. Peloso and A. Ricciardone, Phys. Rev. D **87**, 023504 (2013) [arXiv:1210.3257 [astro-ph.CO]]; A. Maleknejad, M. M. Sheikh-Jabbari and J. Soda, Phys. Rept. **528**, 161 (2013) [arXiv:1212.2921 [hep-th]]; J. Ohashi, J. Soda and S. Tsujikawa, JCAP **1312**, 009 (2013) [arXiv:1308.4488 [astro-ph.CO]].
- [35] D. Giannakis and W. Hu, Phys. Rev. D **72**, 063502 (2005) [astro-ph/0501423]; F. Arroja and M. Sasaki, Phys. Rev. D **81**, 107301 (2010) [arXiv:1002.1376 [astro-ph.CO]]; A. De Felice, S. Mukohyama and S. Tsujikawa, Phys. Rev. D **82**, 023524 (2010) [arXiv:1006.0281 [astro-ph.CO]].
- [36] R. M. Wald, Phys. Rev. D **28**, 2118 (1983).
- [37] A. De Felice and S. Tsujikawa, JCAP **1203**, 025 (2012) [arXiv:1112.1774 [astro-ph.CO]].
- [38] A. Mehrabi, A. Maleknejad and V. Kamali, arXiv:1510.00838 [astro-ph.CO].
- [39] S. Alexeyev, A. Toporensky and V. Ustiansky, Phys. Lett. B **509**, 151 (2001) [gr-qc/0009020]; A. Toporensky and S. Tsujikawa, Phys. Rev. D **65**, 123509 (2002) [gr-qc/0202067].